Analysis Exam, August 2017

1. Let D^* denote the punctured unit disk $\{z \in \mathbf{C} \; : \; 0 < |z| < 1\}$.

(a) Show that if f is holomorphic on D^* and $\lim_{z\to 0} zf(z) = 0$, then the singularity of f at 0 is removable.

(b) Describe the set of all holomorphic g on D^* which satisfy $|g(z)| \leq \log(1/|z|)$. Justify your answer.

2. (a) What is the defining condition for a sequence f_n of L^1 functions on $[0,1]$ to be a *Cauchy sequence* with respect to the L^1 norm?

(b) Give an example of a L^1 Cauchy sequence which consists of continuous functions but which does not converge in L^1 to a continuous function.

(c) Show that any such L^1 Cauchy sequence which consists of characteristic functions does converge in L^1 to a characteristic function.
(Recall a *characteristic function* is one which has only the two values 0 and 1.)

3. Suppose f is holomorphic on the disk $D = \{z : |z| < 1\}$, $\varepsilon > 0$, and $\lim_{n\to\infty} f(z_n) = 0$ for any sequence $z_n \in D$ that converges to $e^{i\theta}$ for some positive $\theta < \varepsilon$. Prove that f is identically 0 .

4. Let K be a nonempty compact subset in $\mathbf{R}^3$ and let $f(x) = \text{dist}\,(x, K)$, and let $g = \max\{1 - f, 0\}$ Prove that $\lim_{n\to\infty} \int_{\mathbf{R}^3} g^n \, dx$ exists and is equal to meas$\,(K)$.

5. Consider a Lebesgue measurable subset E of $\mathbf{R}$ with finite positive measure. On $\mathbf{C}\setminus E$ define the function

$$g(z) \;=\; \int_E \frac{1}{t-z}\, dt \; .$$

(a) Prove that g is holomorphic on $\mathbf{C} \setminus E$.

(b) Prove that g *cannot* be extended to function holomorphic on all of $\mathbf{C}$.

(c) Show that $\lim_{z\to\infty} zf(z)$ exists and determine its value.

6. Suppose μ is a measure on X and E_n is a sequence of μ measurable sets.

(a) Prove the statement:
If $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then μ almost every point $x \in X$ belongs to at most finitely many of the sets E_n.

(b) Is the converse true? That is, does the assumption $\mu(\{x \; : x \text{ belongs to infinitely many } E_n\}) = 0$ imply that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$? Prove the converse if it is true or provide a counterexample if it is false.