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ANALYSIS QUAL WED MAY 11
PROBLEMS BY FRANK
Please comment!

**1.** Let $f \geq 0$ be a measurable function defined on $[0, \infty)$ such that

$$\int_0^x f(t)dt \leq e^x \text{ for all } 0 \leq x < \infty.$$

Suppose that $a > 1$. Prove that

$$\int_0^\infty f(t)e^{-at}dt < \infty.$$

**2.** Suppose that $f$ is a holomorphic function defined on some neighborhood of the origin. Suppose that $f$ satisfies the equation

$$f(2z) = 2f(z)f'(z) \left( = 2f(z)\frac{df}{dz} \right)$$

for all $|z| < \varepsilon$ for some $\varepsilon > 0$. Prove that there exists an <u>entire</u> holomorphic function which is equal to $f$ on some neighborhood of the origin.

**3.** Let $f$ and $g$ be real valued functions belonging to $L^2(\mathbb{R})$ (they are measurable and their squares are integrable on $\mathbb{R}$). Let $h$ be the <u>convolution</u> of $f$ and $g$. That is, $h$ is the function defined by

$$h(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

Prove that $h$ is a bounded continuous function on $\mathbb{R}$, and that $\lim_{|x|\to\infty} h(x) = 0$.

**4.** Let $f$ be the function defined by

$$f(z) = \int_{-1}^{1} \frac{dt}{t-z},$$

where $z \in \mathbb{C}$ and $z$ is not a real number in the interval $[-1,1]$.

a. Prove that $f$ is holomorphic.

b. Compute $\lim_{\substack{y \to 0 \\ y > 0}} f(iy)$ and $\lim_{\substack{y \to 0 \\ y < 0}} f(iy)$.

**5.** Construct a real valued $C^{\infty}$ function on $\mathbb{R}$ which equals $0$ on $(-\infty, 0]$ and equals $1$ on $[1, \infty)$.

**6.** Suppose $f$ is a holomorphic function defined on the unit disk $|z| < 1$, and suppose $f$ is not identically zero. Is it possible that for every $z_0$ such that $|z_0| = 1$ there exists a sequence $\{z_1, z_2, z_3, \cdots\}$ such that $|z_n| < 1$ for all $n \geq 1$ and $f(z_n) = 0$ for all $n \geq 1$?

Explain.