Qualifying Examination
Analysis

Solutions

May 9, 2016

Problem 1 *Let $w_0 \in \mathbb{C}$. Prove there exists a holomorphic function, $f(\cdot)$, on the disc*

$$\Delta = \left\{ z \middle| \; |z| < e^{-\operatorname{Re} w_0} \right\}$$

such that

$$f(0) = w_0 \qquad (1.1)$$

$$f'(z) = e^{f(z)}, z \in \Delta.$$

Express the solution f in closed form.

Solution #1

One sees that $f''(z) = f'(z)\, e^{f(z)} = e^{2f'(z)}$, and then, by induction on n,

$$f^{(n)}(z) = (n-1)! e^{nf(z)}, n \geq 1$$

$$f^{(n)}(0) = (n-1)! e^{nw_0}, n \geq 1.$$

The Taylor expansion of f about 0 is

$$f(z) = w_0 + \sum_{n=1}^{\infty} \tfrac{f^{(n)}(0)}{n!} z^n \qquad (1.2)$$

$$= w_0 + \sum_{n=1}^{\infty} \tfrac{1}{n} \left(e^{w_0} z\right)^n.$$

The series converges for $|e^{w_0}z| = |z|\, e^{\operatorname{Re} w_0} < 1.$ or

$$|z| < e^{-\operatorname{Re} w_0}.$$

The second line of (1.2) is

$$f(z) = w_0 + \log \frac{1}{1 - e^{w_0 z}}. \tag{1.4}$$

Solution #2

The second line of (1.1) is the same as

$$f'(z)\, e^{-f(z)} = 1.$$

If there is a solution on Δ, it must satisfy

$$z = \int_0^z f'(w)\, e^{-f(w)} dw$$

$$= -e^{-f(w)}\big|_0^z = e^{-w_0} - e^{-f(z)}$$

Rearrangement yields the same function (1.4).

Problem 2 *Let $0 < r < 1$, and denote by C_r the circle of radius r, with the positive (counterclockwise) orientation. Prove that*

$$\int_{C_r} \frac{1}{(1-\overline{z})^{n+1}} dz = 2\pi i\,(n+1)\, r^2, n \geq 0.$$

What about $n < 0$?

Solution

If $z \in C_r$, then $\overline{z} = \frac{r^2}{z}$, and the integral may be rewritten

$$\int_{C_r} \frac{1}{(1-\overline{z})^{n+1}} dz = \int_{C_r} \frac{z^{n+1}}{(z-r^2)^{n+1}} dz.$$

Since $r^2 < r < 1$, the Cauchy integral formula for the n^{th} derivative implies

$$\int_{C_r} \frac{z^{n+1}}{(z-r^2)^{n+1}} dz = (2\pi i) \frac{\left(z^{n+1}\right)^{(n)}\left(r^2\right)}{n!}$$

$$= 2\pi i (n+1) r^2.$$

If $n < 0$, then $\frac{1}{(1-\overline{z})^{n+1}} = (1-\overline{z})^{-n-1}$ is a polynomial in $\overline{z}$,

$$(1-\overline{z})^{-n-1} = \sum_{k=0}^{-n-1} \binom{-n-1}{k} (-\overline{z})^k.$$

The only power with a nonzero integral is the first, and therefore

$$\int_{C_r} (1-\overline{z})^{-n-1} dz = (-n-1) \int_{C_r} -\overline{z} dz = 2\pi i (n+1) r^2.$$

Problem 3 *Let $f(\cdot)$ be holomorphic on a neighborhood, U, of a point z_0. Define $g(\cdot)$ on U by*

$$g(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0}, z \in U, z \neq z_0 \\ f'(z_0), z = z_0 \end{cases} \tag{3.1}$$

*Give **TWO** proofs that $g(\cdot)$ is holomorphic on U. For each proof, cite carefully theorem(s) about holomorphic functions that imply the statement under consideration*

Solution #1

Since $\lim_{z \to z_0} g(z) = g(z_0)$, $g(\cdot)$ is bounded and holomorphic on a punctured disc about z_0. The Riemann Removable singularities theorem says that g extends to be holomorphic on the unpunctured disc. Since the extension is, in particular, continuous, the function (3.1) is holomorphic.

Solution #2

Since $f(\cdot)$ is holomorphic on U, the Taylor series of f converges locally uniformly to f on a disc, Δ, about z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n .$$

Therefore, if $z \neq z_0$,

$$\frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) + \sum_{n=2}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^{n-1} .$$

The power series converges on Δ and is zero at z_0. Therefore, $g(\cdot)$ is holomorphic.

Problem 4 *Let $f \geq 0$ be real-valued and measurable on $\mathbb{R}$. Denote by m the Lebesgue measure on $\mathbb{R}$. Prove that, whether or not f is integrable,*

$$\lim_{N\to\infty} \sum_{k=0}^{\infty} \frac{k}{N} m \left(f^{-1} \left(\left(\frac{k}{N}, \frac{k+1}{N} \right] \right) \right) = \int_{\mathbb{R}} f(x)\, m(dx) . \tag{4.1}$$

Solution

Let f_N be the function

$$f_N(x) = \sum_{k=0}^{\infty} \frac{k}{N} \chi_{f^{-1}\left(\left(\frac{k}{N}, \frac{k+1}{N}\right]\right)}(x) \leq f(x) . \tag{4.2}$$

Since f takes values in $\mathbb{R}$, it is true that for all x

$$\lim_{N\to\infty} f_N(x) = f(x) , x \in \mathbb{R}. \tag{4.3}$$

The Monotone Convergence Theorem implies for each N that

$$\int_{\mathbb{R}} f_N(x)\, m(dx) = \lim_{L\to\infty} \int_{\mathbb{R}} \sum_{k=0}^{L} \tfrac{k}{N} \chi_{f^{-1}\left(\left(\frac{k}{N}, \frac{k+1}{N}\right]\right)}(x)\, m(dx)$$

$$= \lim_{L\to\infty} \sum_{k=0}^{L} \tfrac{k}{N} m\left(f^{-1}\left(\left(\tfrac{k}{N}, \tfrac{k+1}{N}\right]\right)\right) = \sum_{k=0}^{\infty} \tfrac{k}{N} m\left(f^{-1}\left(\left(\tfrac{k}{N}, \tfrac{k+1}{N}\right]\right)\right) . \tag{4.4}$$

By (4,2)-(4.4) and Fatou's Lemma

$$\int_{\mathbb{R}} f(x)\, m(dx) \geq \limsup_{N\to\infty} \int_{\mathbb{R}} f_N(x)\, m(dx)$$

$$\geq \liminf_{N\to\infty} \int_{\mathbb{R}} f_N(x)\, m(dx) \geq \int_{\mathbb{R}} f(x)\, m(dx)$$

Problem 5 *Let f be a function on the (x, y) plane. Assume that for each fixed x (resp. for each fixed y) $f(x, y)$ is continuous in y (resp. $f(x, y)$ is continuous in x). Prove that f is a Borel function on the plane. Hint: You may wish to consider the functions*

$$f_{m,n}(x, y) = f\left(\frac{[mx]}{m}, \frac{[ny]}{n}\right), m, n > 0.$$

($[w] = \max\{n \in \mathbb{Z} | n \leq w\}$ is the greatest integer function.)

Solution

For all $y \in \mathbb{R}$ it is true for $n > 0$ that

$$[ny] \leq ny < [ny] + 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{[ny]}{n} = y.$$

The functions $f_{m,n}(x, y)$ are Borel (constant on each rectangle, $R_{m,n}(k, l)$,

$$R_{m,n}(k, l) = \left[\frac{k}{m}, \frac{k+1}{m}\right) \times \left[\frac{l}{n}, \frac{l+1}{n}\right), k, l \in \mathbb{Z},$$

of a partition of $\mathbb{R}^2$ into rectangles). By the assumption on f,

$$\lim_{n\to\infty} f_{m,n}(x, y) = f\left(\frac{[mx]}{m}, y\right), \text{ all } m > 0, x, y \in \mathbb{R}.$$

Since a pointwise everywhere limit of a sequence of Borel functions is Borel, the functions

$$f_m(x, y) = f\left(\frac{[mx]}{m}, y\right)$$

are Borel for all m. Once again, the assumption on f implies

$$\lim_{m\to\infty} f_m(x,y) = f(x,y)$$

and $f(x,y)$ is Borel.

REMARK

On the Continuum Hypothesis, there exists on the plane a function that is **not even Lebesgue measurable** and which has the property that restricted to any line in the plane it has at most two discontinuities. (Notes attached.)

Problem 6 *let f be measurable on $\mathbb{R}$, and assume that both $f(x)$ and $xf(x)$ are integrable with respect to Lebesgue measure, m. Define $F(y)$ on $\mathbb{R}$ by*

$$F(y) = \int_{\mathbb{R}} f(x) \sin xy \; m(dx).$$

Prove that F is differentiable on $\mathbb{R}$ and that

$$F'(y) = \int_{\mathbb{R}} xf(x)\, cos(xy)\, m(dx). \tag{6.1}$$

Solution

For all x, y and $\delta \neq 0$ the mean value theorem implies there is a point $\xi(x,y,\delta)$ strictly between y and $y+\delta$ such that

$$\left|\frac{\sin x[y+\delta] - \sin xy}{\delta}\right| = |x\cos(x\xi(x,y,\delta))|$$

$$\leq |x|$$

Now, use the assumption that $xf(x)$ is Lebesgue integrable: As

$$\frac{F'(y+\delta) - F'(y)}{\delta} = \int_{\mathbb{R}} f(x) \frac{\sin x(y+\delta) - \sin xy}{\delta} m(dx)$$

and

$$\left| f(x) \frac{\sin x[y+\delta] - \sin xy}{\delta} \right| \leq |xf(x)|,$$

the dominated convergence theorem implies that (6.1) is true.

REMARK ON PROBLEM 5:

A REASON TO OPPOSE

THE CONTINUUM HYPOTHESIS

Let $(E, \leq) = E$ be a nonempty partially ordered set. E is **well-ordered** if every nonempty subset of E has a least element. Notice that if $x, y \in E$, then either $x = y$ or one of the inequalities $x < y$, $y < x$ must hold. That is, a well-ordered set is **totally ordered.** An induction argument shows that any nonempty **finite** subset of a well-ordered set has a **maximum** element.

If E is well-ordered, if $x \in E$, and if

$$S(x) \stackrel{def}{=} \{y \in E|\ x < y\} \neq \emptyset.$$

then $S(x)$ has a least element, which we denote by

$$\sigma(x) = \min S(x).$$

Refer to $\sigma(x)$ as the **successor** of x. There are two kinds of elements of E, successors and limit elements, i.e., elements which have no immediate predecessors. Example,

$$E = \{0, 1, 2, \cdots, \infty\}.$$

with the natural order. ∞ is not a successor.

If $(E, \leq)$ is nonempty and well-ordered, there exists a least element of E. Define

$$0 = \min E.$$

For any $x \in E$ define an "interval" $[0, x)$ by

$$[0, x) = \{y|\ y < x\}.$$

Proposition 7 *Let $(E, \leq) = E$ be an uncountable well-ordered set. Define $E^{\omega} \subseteq E$ by*

$$E^{\omega} = \{x|\ [0, x) \text{ is countable}\}.$$

Then E^{ω} is an uncountable "initial segment" of E. That is, E^{ω} is an uncountable set, and if $x \in E^{\omega}$, then $[0, x) \subset E^{\omega}$.

Proof. If E^ω were countable, the assumption that E is uncountable implies the set

$$\widehat{E} = E \backslash E^\omega \neq \emptyset.$$

Since it is nonempty, $\widehat{E}$ has a smallest element. Let $z = \min \widehat{E}$. If $x < z < y$, then by definition $x \in E^\omega$. Since $[0, z) \subset [0, y)$ and $[0, z)$ is uncountable, $[0, y)$ is uncountable. In particular, $y \notin E^\omega$. Therefore, $E^\omega = [0, z)$, contradicting the assumption that E^ω is countable. ■

The proposition implies that an uncountable well-ordered set E contains an uncountable well-ordered subset with the property that each of its initial segments is **countable.** From now on we assume fixed

$$(E, \leq) \text{ an uncountable well-ordered set}$$

$$\text{such that} \tag{CH1}$$

$$[0, x) \text{ is countable, all } x \in E.$$

To see that such a set exists, begin with any uncountable set, F, e.g., the set of all subsets of the integers, apply the axiom of choice to well order F, and then use the discussion above to see that $E = F^\omega$ has the desired property.

We now state one assumption and one known fact:

ASSUMPTION:

The continuum hypothesis is true. The set (CH1) has (CH2)

the same cardinality as the set of real numbers.

FACT

The set, $\mathcal{B}(\mathbb{R}^2)$, of Borel sets has the same cardinality (CH3)

as the set of real numbers.

A PDF of a proof of (CH3), which does not require the continuum hypothesis, is available upon request to interested students.

Theorem 8 *There exists a set* $\Omega \subset \mathbb{R}^2$ *such that*

1. For any line $L \subset \mathbb{R}^2$, $L \cap \Omega$ contains at most two points.

2. Ω contains no uncountable Borel set.

3. If $B \subset \mathbb{R}^2 \backslash \Omega$ is a Borel set, then B is contained in a countable union of lines.

4. Ω is not Lebesgue measurable.

Proof. Denote by $\mathcal{B}_c(\mathbb{R}^2)$ the set of uncountable Borel sets. By (CH2) and (CH3), there is a one-to-one and onto map,

$$E \rightarrow \mathcal{B}_c(\mathbb{R}^2)$$

$$x \rightarrow B_x \in \mathcal{B}_c(\mathbb{R}^2), \ x \in E.$$

We shall make an inductive construction of points

$$p(x), q(x), \ x \in E.$$

First, choose for $x = 0 = \min(E)$

$$p(0), q(0) \in B_0, \ p(0) \neq q(0).$$

Now assume $0 < y \in E$ and that the construction has been made for $0 \leq x < y$ with properties now to be described. Define

$$\Omega(y) = \{p(x) | \ x < y\}$$

$$\Lambda(y) = \{q(x) | \ x < y\}.$$

Assume

A. For any line $L \subset \mathbb{R}^2$, $L \cap \Omega(y)$ contains at most two points.

B. $\Lambda(y) \cap \Omega(y) = \emptyset$.

C. If $x < y$ is such that B_x is not contained in a countable union of lines, then $p(x) \in B_x$. Otherwise, $p(x) = p(0)$.

D. If $x < y$, then $q(x) \in B_x$.

We shall construct the values $p(y), q(y)$. $\Omega(y)$ is a countable set because $[0, y)$ is countable for each $y \in E$. Therefore, the set of pairs of distinct points in $\Omega(y)$ determines a countable set of lines. Denote the union of these lines by $\Gamma(y)$. If B_y is contained in a countable union of lines, define $p(y) = p(0)$. If B_y is not contained in a countable union of lines, $B_y \backslash \Gamma(y)$ cannot be a countable set. For $B_y \backslash \Gamma(y)$ were countable, it would also determine a countable set of lines whose union with $\Gamma(y)$ would contain B_y. Now choose $p(y) \in B_y \backslash \Gamma(y)$. Having chosen $p(y)$, define

$$\Omega(\sigma(y)) = \Omega(y) \cup \{p(y)\}.$$

Now to choose $q(y)$. Let $q(y)$ be any point in the uncountable set $B_y \backslash \Omega(\sigma(y))$. Let

$$\Lambda(\sigma(y)) = \Lambda(y) \cup \{q(y)\}.$$

The induction hypothesis and the fact $p(y) = p(0)$ or $p(y) \in B_y \backslash \Gamma(y)$ imply that no line contains more than two points of $\Omega(\sigma(y))$.This is Property A. Since $p(y) \neq q(y)$ and $p(y) \notin \Lambda(y), q(y) \notin \Omega(y)$, it is true that Property B holds for $\sigma(y)$. The induction hypothesis and the choices of $p(y)$ and $q(y)$ imply Properties C. and D. for $\sigma(y)$. The construction is now completed by induction. Define $\Omega, \Lambda \subset \mathbb{R}^2$ by

$$\Omega = \{p(y) | \ y \in E\}$$

$$\Lambda = \{q(y) | \ y \in E\}$$

$$\Lambda \cap \Omega = \emptyset.$$

The last line is due to B. If $B \in \mathcal{B}_c(\mathbb{R}^2)$, then $B = B_y$ for some $y \in E$. By D. $q(y) \in \Lambda(\sigma(y)) \subset \Lambda \subset \mathbb{R}^2 \backslash \Omega$. Therefore, $B \nsubseteq \Omega$, and 2. is true. Suppose that $B \in \mathcal{B}_c(\mathbb{R}^2)$ is not contained in a countable union of lines. If $B = B_y$, then by C. $p(y) \in B$. Therefore, $B \nsubseteq \Lambda$, and 3. is true. If $L \subset \mathbb{R}^2$ is a line, and if $L \cap \Omega$ contains at least three points, there would exist $y \in E$ such that $L \cap \Omega(y)$ contains at least three points, contradicting A. Therefore, Ω satisfies 3. Finally, if Ω is Lebesgue measurable, Ω must be a null set. For otherwise, the inner regularity of Lebesgue measure would imply that Ω contains a compact, in particular Borel, set of positive measure, in violation of 2. If Ω is a null set, then Λ is a Lebesgue measurable set of positive (infinite!) measure. Again applying inner regularity, Λ would contain a Borel set of positive measure. Such a set cannot be contained in a countable union of lines, in violation of 3. We conclude that Ω is not a Lebesgue measurable set. ■