Analysis Exam, May 2017

1. Suppose $f_n : \mathbf{R} \to \mathbf{R}$ is a differentiable function for $n = 1, 2, \ldots$,
$M = \sup_{n,x} |f_n'(x)| < \infty$, and $f(x) = \lim_{n\to\infty} f_n(x) \in \mathbf{R}$ exists for all $x \in \mathbf{R}$.

(a) Is f *continuous* on $\mathbf{R}$? Prove or find a counterexample.

(b) Is f *differentiable* on $\mathbf{R}$? Prove or find a counterexample.

(c) Does $\int_0^1 f(x)\,dx \;=\; \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$? Prove or find a counterexample.

2. (a) Find all the poles of the function $f(z) = \frac{1}{z \cos z \cosh z}$.

(b) Calculate the residue of f at each pole.

3. Suppose f is a nonnegative Lebesgue measurable function on $\mathbf{R}$.
Recall that, for any $1 \le p < \infty$, the numbers (possibly $+\infty$)

$$\|f\|_p \;=\; \Big(\int_{\mathbf{R}} f^p(x)\,dx\Big)^{1/p} \quad \text{and} \quad \|f\|_\infty \;=\; \text{essential supremum of } f \;.$$

(a) Suppose that $\|f\|_1 < \infty$. Prove that

$$\lim_{p\uparrow\infty} \|f\|_p \;=\; \|f\|_\infty \;. \tag{$*$}$$

(b) Give an example of an f with $\|f\|_1 = \infty$ so that (*) is false.

4. Compute

$$\int_0^\infty \frac{\log x}{x^2 - 1}\,dx \;.$$

5. (a) Show that a sequence $f_1, f_2, f_3, \ldots$ of real-valued functions on $\mathbf{R}$ converges uniformly

$$\text{if and only if} \qquad \lim_{n\to\infty} \sup_{m>n} \sup_{x\in\mathbf{R}} |f_m(x) - f_n(x)| \;=\; 0 \;.$$

(b) Suppose $P_1, P_2, P_3, \ldots$ is a sequence of real-valued polynomials on $\mathbf{R}$ and

$$\lim_{n\to\infty} P_n(x) \;=\; Q(x) \qquad \text{uniformly on } \mathbf{R} \;.$$

What can you say about the limit function Q ?

6. Let f be a not-identically-zero holomorphic function from the upper half plane $\mathcal{I}m\, z > 0$ into the unit disk $|z| < 1$. Also suppose $f(\mathbf{i}) \;=\; 0$.

How big and how small can $|f(2\mathbf{i})|$ be ?