Analysis Exam, January 2013

1. Suppose a and b are two points on the unit circle, and f is a nonconstant holomorphic function on the unit disk D.

(a) Show that

$$\lim_{t \downarrow 0} \frac{|f(ta)|}{|f(tb)|} \;=\; 1\ .$$

(b) Is this still true if f is meromorphic with a pole at 0?

2. Let $C[0,1]$ be the space of continuous functions on the closed interval $[0,1]$, and

$$d(f,g) \;=\; \int_0^1 |f(t)-g(t)|\,dt \quad \text{for} \quad f,\, g \in C[0,1]\ .$$

(a) Prove that d defines a *metric* on $C[0,1]$.

(b) Is $C[0,1]$, with the metric d, a *complete metric space*? Prove your answer.

3. Suppose $A = \{z \in \mathbf{C} \ :\ 1 < |z| < \infty\}$.

(a) Does there exist a harmonic function $h : A \to \mathbf{R}$ so that $\lim_{|z|\to\infty} h(z) = \infty$? If so, find an example. If not, explain why not.

(b) Does there exist a nonconstant holomorphic function $f : A \to \mathbf{C}$ so that $\lim_{|z|\to\infty} Re(f(z)) = \infty$? If so, find an example. If not, explain why not.

4. Let E be a Lebesgue measurable subset of $\mathbf{R}$ with finite measure, and let

$$F(t) \;=\; \int_E \sin(tx)\,dt$$

(a) Prove that F is continuous.

(b) Prove that F is differentiable if E is bounded.

5.(a) Determine all holomorphic functions f from the unit disk to itself with $f(0) = 0$ and $|f(1/2)| = 1/2$.

(b) Determine all holomorphic functions g from the unit disk to itself with $g(1/2) = 1/2$ and $g(-1/2) = -1/2$.

6. (a) Give a definition of a *Lebesgue measurable function on* $\mathbf{R}$.

(b) Show that if $f_1, f_2, , \ldots$ is a sequence of continuous functions on $\mathbf{R}$ and $f(x) = \lim_{k\to\infty} f_k(x)$ for almost every $x \in \mathbf{R}$, then f is Lebesgue measurable.

(c) Show that, conversely, every Lebesgue measurable function f on $\mathbf{R}$ is the pointwise almost everywhere limit of some sequence of continuous functions.