RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - AUGUST 2011

This is a 3-hour closed book, closed notes exam. Please show all of your work.

1. Suppose X is a path-connected Hausdorff space. Let $p : \widetilde{X} \to X$ be a covering map. Prove that if $\widetilde{X}$ is compact then p is a finite-sheeted covering space.

2. Suppose S is a compact, connected surface without boundary. Suppose $\pi_1(S)$ has a *proper* finite index subgroup that is isomorphic to $\pi_1(S)$. For which surfaces is this possible? (Prove it)

3. Let X be the space obtained from a solid octagon by identifying sides as shown in Figure 1 below.

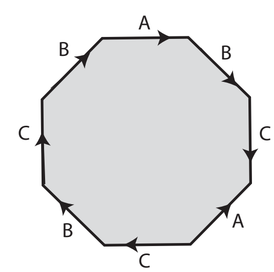

FIGURE 1

(a) Give a CW-structure for X (be careful with the vertices) and describe the cellular chain complex.
(b) Give a presentation for $\pi_1(X)$ (be careful with the vertices) .
(c) Calculate $H_n(X; \mathbb{Z}_3)$ and $H^n(X; \mathbb{C})$ for all $n \geq 0$.

4. Give an example (a CW complex) for each of the following or state "such an example does not exist because...". Give brief justifications in all cases.

a) two spaces with isomorphic π_1 but non-isomorphic integral homology groups;

b) two spaces with isomorphic integral homology groups but non-isomorphic π_1; Give π_1 of the spaces.

c) two spaces with isomorphic integral homology groups but non-isomorphic cohomology groups.

d) two spaces that are homotopy equivalent but not homeomorphic.
e) two spaces with isomorphic π_1 and isomorphic integral homology groups that are NOT homotopy equivalent.

5. Let X be a connected, orientable, compact 4-dimensional manifold without boundary such that $\pi_1(X) \cong \mathbb{Z}_{35}$ and $\chi(X) = 4$.
(a) Calculate $H_i(X; \mathbb{Z})$ for all i.
(b) Suppose $\widetilde{X}$ is a connected regular 7-fold (7-sheeted) covering space of X. Calculate $H_i(\widetilde{X}; \mathbb{Z})$ for all i.