Rice University Topology Qualifying Exam

August, 2014

This is a 3 hour, closed book, closed notes exam. For maximum credit include justification for all steps. Sign the Rice honor pledge at the end of your exam.

1. Let Σ be a closed oriented genus g surface.
   (a) Let γ be a simple closed nonseparating curve on Σ. What surface results from cutting Σ along γ? Make sure to justify your answer!
   (b) Let γ_1 and γ_2 be two simple closed nonseparating curves on Σ. Prove that there exists some homeomorphism $f\colon \Sigma \to \Sigma$ such that $f(\gamma_1) = \gamma_2$. Hint: Use part a.

2. Let X be a connected CW complex such that $\pi_1(X)$ is a finite group. Prove that any continuous map $f\colon X \to S^1$ is homotopic to a constant map. Hint: covering spaces! What is the image of $f_* : \pi_1(X) \to \pi_1(S^1)$?

3. Let $K = D^2 \times S^1$ be a solid torus and let $f\colon K \to S^3$ be an arbitrary embedding. Define $X = S^3 \setminus \mathrm{Int}(f(K))$. Prove that $\mathrm{H}_1(X;\mathbb{Z}) \cong \mathbb{Z}$.

4. Let $\{p_1, \ldots, p_k\}$ be a set of distinct points in S^2. Define X to be the quotient space of S^2 that identifies all of the p_i to a single point. Calculate $\pi_1(X)$ and $\mathrm{H}_i(X;\mathbb{Z})$ for all i.

5. Recall that if M and N are compact connected oriented k-manifolds, then the degree of a map $g\colon M \to N$ is the number d such that the induced map $g_*\colon \mathrm{H}_k(M;\mathbb{Z}) \to \mathrm{H}_k(N;\mathbb{Z})$ takes $x \in \mathrm{H}_k(M;\mathbb{Z}) \cong \mathbb{Z}$ to $dx \in \mathrm{H}_k(N;\mathbb{Z}) \cong \mathbb{Z}$. Prove that if X is a compact connected oriented p-manifold and Y is a compact connected oriented q-manifold, then every continuous map $f\colon S^{p+q} \to X \times Y$ has degree 0.

6. Let M be a compact connected manifold with ∂M connected and nonempty. Prove that M does not retract onto ∂M, i.e. there does not exist a map $f\colon M \to \partial M$ such that the composition
$$\partial M \hookrightarrow M \xrightarrow{f} \partial M$$
is the identity.