Topology Qualifying Exam
Rice University - August 2015

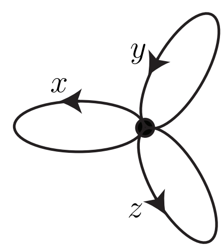
This is a 4 hour, closed book, closed notes exam. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Prove that $SL_n(\mathbb{R})$ is a smooth manifold and calculate its dimension (with proof).

2. Let $F(n)$ be the free group of rank n. For each integer $n \geq 2$, prove that $F(2)$ contains a finite index normal subgroup isomorphic to $F(n)$.

3. Let $W = S^1 \vee S^1 \vee S^1$ be as shown below. Let x, y, z be the three loops indicated going around the first, second, third circles respectively. Let $X = W \cup_{f_1} e_1^2 \cup_{f_2} e_2^2$ be the space obtained from W by adjoining one 2-cell via the map f_1 which forms the loop $xyx^{-1}zy^{-1}z^{-1}$; and another 2-cell via the map f_2 which forms the loop x^7.



(a) Write down a cellular chain complex for X (including the boundary maps).

(b) Use (a) to compute $H_p(X;\mathbb{Z}_2)$ and $H_p(X;\mathbb{Z})$ for all p. Do not use a Universal Coefficient Theorem.

(c) Use the Universal Coefficient Theorem for Cohomology to compute $H^p(X;\mathbb{Q})$ for all p.

4. Let $T = S^1 \times S^1$ and let $f : T \to T$ be defined by

$$f(x, y) = (2x + y, x + y).$$

Here we are viewing S^1 as $\mathbb{R}/\mathbb{Z}$. Let $X = (T \times [0, 1])/\sim$ be the 3-manifold obtain by identifying $(x, y) \times \{0\}$ with $f(x, y) \times \{1\}$. Compute $\pi_1(X)$.

5. Let M be a closed, connected, orientable 4-dimensional manifold with $\pi_1(M) \cong \mathbb{Z}_5 * \mathbb{Z}_5$ and $\chi(M) = 5$.

(a) Compute $H_p(M; \mathbb{Z})$ for all p.

(b) Prove that M is not homotopy equivalent to a CW complex with no 3-cells.

6. Let $n \geq 0$ be an even integer. Prove that there is no orientation-reversing (that is degree -1) map $f : \mathbb{C}P(n) \to \mathbb{C}P(n)$.