Rice University Topology Qualifying Exam May 2009

Justify all of your work. No books or notes allowed. 3 hours time limit.

1. Suppose Y is the space obtained from a wedge of two circles (labelled x and y) by adjoining two 2-cells, the first via a map $f : \partial D^2 \to S^1 \vee S^1$ given by $xyxy^{-1}$ and the second via a map $g : \partial D^2 \to S^1 \vee S^1$ given by $yxyxy^{-2}$.
a. give a presentation for $\pi_1(Y)$.
b. Give the cellular chain complex of Y.
c. Compute $H_p(Y;\mathbb{Z})$ for all p.
d. Compute $H^p(Y;\mathbb{Z}_3)$ for all p.

2. Suppose H is the subgroup of the free group $F(x,y)$ (free on the set $\{x,y\}$) that is generated by four elements as follows $H = \langle x^2,\ y^2,\ xy^2x,\ x^{-1}y\rangle$.
a. Is H normal in $F(x,y)$? Explain why/why not.
b. What is the index of H in $F(x,y)$? Show it.
c. Construct a covering space of the wedge of two circles corresponding to the subgroup H. Explain why this is the correct covering space.
d. Is H a free group? Why or why not? If it is a free group, write down a free basis for H.
e. Is $xyx \in H$? Prove it.

3. Let T be the torus, $T \equiv S^1 \times S^1$. Let M be the Möbius band. Form an identification space $X = T \cup M$ by identifying the *boundary circle* of the Mobius band to the circle $S^1 \times \{1\} \subset T$.
a. Compute a presentation for $\pi_1(X)$.
b. Compute $H_p(X,\mathbb{Z})$ for all p.
c. Is X a manifold? Why or why not (briefly)?
d. Compute $H_p(X,M;\mathbb{Z})$ for all p.

4. Prove: if $n > 1$ then any continuous map $f : S^n \to K$, where K is the Klein Bottle, is null-homotopic.

5. Let M be a closed, connected, orientable 4-dimensional manifold with $\pi_1(M) \cong \mathbb{Z}_5 * \mathbb{Z}_5$ and $\chi(M) = 5$.
a. Compute $H_p(M;\mathbb{Z})$ for all p.
b. Prove that M is not homotopy equivalent to a CW complex with no 3-cells.

6.
a. Describe the cohomology ring of $\mathbb{C}P(n)$. Briefly sketch how this is proved.
b. Prove that the degree of any continuous map $f : \mathbb{C}P(n) \to \mathbb{C}P(n)$ is m^2 for some integer m.