Rice University Topology Qualifying Exam Jan 2009

Show all your work. No books or notes allowed. 3 hours time limit.

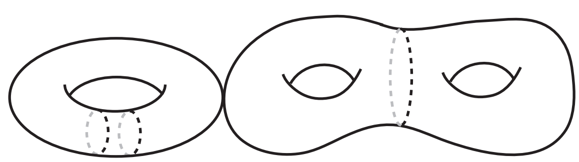
1. Prove: If $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ are covering maps then $p \times q : \tilde{X} \times \tilde{Y} \to X \times Y$, given by $(a, b) \mapsto (p(a), q(b))$, is also a covering map.

2. a. Calculate $H_p(\mathbb{R}P(n); \mathbb{Z}_4)$ for all $p \geq 0$.

b. Calculate, **without using a Universal Coefficient Theorem**, $H^p(\mathbb{R}P(n); \mathbb{Q})$ for all $p \geq 0$, where $\mathbb{R}P(n)$ is n-dimensional real projective space.

3. If X= a 2-dimensional disk with two open sub-disks removed, discuss **2-fold** covering spaces of X. Your discussion might include: relations to $\pi_1(X)$, examples, pictures, what are the possible groups of covering transformations, how many covers are there up to equivalence, are they regular?

4. Suppose Y is the space obtained from the wedge of a torus and a genus two surface (pictured below),by adjoining three 2-dimensional disks, two along meridional circles of the torus, and the third along the "waist" circle of the genus two surface. The three attaching circles of these disks are the dashed circles. The three disks are not pictured.



a. Show that Y has the structure of a CW complex.

b. Compute $\pi_1(Y)$.

c. Compute $H_p(Y; \mathbb{Z})$ for all p.

5. Let $X = S^1 \times \mathbb{C}P(2)$.

a. Calculate $\pi_1(X)$.

b. Note that $\mathbb{C}P(2)$ is a retract of X. What implications does this have for the homology groups of X ?

c. Prove that there is a class $\gamma \in H^2(X; \mathbb{Z})$ such that $\gamma \cup \gamma$ is non-zero.

d. Assuming that X is orientable, compute $H_p(X; \mathbb{Z})$ for all p. (Hint: it might be useful to know its Euler characteristic or to use parts a. and b.).