Analysis Exam, May 2008

1. Compute the following limits, justifying all your steps.

(a)

$$\lim_{n\to\infty} \int_0^{e^n} \frac{x}{1+nx^2}\, dx$$

(b)

$$\lim_{n\to\infty} \int_0^{n} \frac{\sin nx}{1+nx^2}\, dx\ .$$

2. Determine the number of zeros of $f(z) = z^6 + z^3 + 5z^2 - 2$, appropriately taking into account multiplicity, in the annulus $\{z \in \mathbf{C}\ :\ 1 < |z| < 2\}$. Justify your conclusion.

3. Suppose $1 \le p < q < r \le \infty$. Prove that if $f \in L^p(\mathbf{R}) \cap L^r(\mathbf{R})$, then $f \in L^q(\mathbf{R})$.

4. Suppose that f, g, and h are holomorphic functions on $\mathbf{C} \setminus \{0\}$.

(a) Find, if possible, such an f where the derivative f' has a pole of order 1 at 0. If this is impossible, explain why.

(b) Suppose $\lim_{|z|\to\infty} z^{-4}g(z) = 2$. What can one say about $\lim_{|z|\to\infty} z^{-3}g'(z)$?

(c) Assuming $\lim_{|z|\to\infty} z^{-4}h(z) = 2$ and $\lim_{|z|\to 0} |z^3 h(z)| = 3$, what is the general form for h?

5. Suppose f is a continuous function on $\mathbf{R}$ and the derivative f' exists, is bounded, and is continuous almost everywhere. Let $g(x) = \int_0^x f'(t)\, dt$.

(a) Is it true that $g'(x) = f'(x)$ for ~~every~~ almost every $x \in \mathbf{R}$? Prove this if it is true or find a counterexample if it is false.

(b) Is it true that $g(x) = f(x) - f(0)$ for almost every $x \in \mathbf{R}$? Prove this if it is true or find a counterexample if it is false.

6. Let D be the unit disk $\{z \in \mathbf{C}\ :\ |z| < 1\}$, $a \in D$, $|a| < 1$, and

$$L_a(z) \;=\; \frac{z-a}{1-\bar{a}z} \quad \text{for } z \in \bar{D}\ .$$

(a) Show that $|L_a(z)| = 1$ whenever $|z| = 1$.

(b) Derive the general formula for a **bijective** holomorphic map $f\ : D \to D$.

(c) Derive the general formula for a (not necessarily injective) holomorphic map $g : D \to D$ such that $\lim_{|z|\to 1} |g(z)| = 1$.