ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, FALL 2023

Instructions:

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: August 16, 2023.

(1) (a) Prove that A_4 (the alternating group on 4 elements) and D_{12} (the dihedral group of order 12) are not isomorphic as groups.

(b) Prove that there exists a non-abelian group of order 12 that is not isomorphic to either A_4 nor D_{12}.

(2) Let $R = \mathbb{Q}[s, x, y, z]$ be a polynomial ring with lexicographic order $s > x > y > z$. Let $I \subset R$ be the ideal

$$I = \langle x - s^3, y - s^2, z - s\rangle.$$

(a) Show that $\{s - z, x - z^3, y - z^2\}$ is a Gröbner basis for I.

(b) Deduce that the kernel of the ring homomorphism

$$\phi\colon \mathbb{Q}[x, y, z] \to \mathbb{Q}[s]$$

determined by $\phi(x) = s^3$, $\phi(y) = s^2$, and $\phi(z) = s$ is *equal* to the ideal $\langle x - z^3, y - z^2\rangle$.

(3) Decide which of the following groups are isomorphic to the trivial group. Provide reasoning.

(a) $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$.

(b) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

(c) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$.

(d) $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$.

(e) $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} 2\mathbb{Z}$.

(4) Let $t = \sqrt{(1 + \sqrt{5})/2}$.

(a) Compute the minimal polynomial $p(x)$ of t.

(b) What is the splitting field of $p(x)$?

(c) What is the Galois group of $p(x)$?

(5) Let $R \subseteq S$ be an inclusion of integral domains with unit, such that S is integral over R. Prove that R is a field if and only if S is a field.

(6) Let R be a commutative ring with 1, let I be an ideal of R, and let M be an R-module. Show that if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of R containing I, then $M = IM$.