ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2023

Instructions:

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: May 11, 2023.

(1) (a) Let p be a prime. Prove that every group of order p^2 is abelian.
(b) Determine the number of isomorphism classes of groups of order 45.

(2) Suppose that R is a Noetherian ring and $I \subset R$ an ideal such that $I^2 = I$. Show that $R \cong I \oplus J$, where $J \subset R$ is another ideal.

(3) Take a matrix $M \in \mathrm{Mat}_4(\mathbb{C})$ with minimal polynomial $m(x) = x^4 - x^3 - x^2 + x$. What is the characteristic polynomial and minimal polynomial of the square M^2 of M?

(4) Suppose $P(x) \in \mathbb{Z}[x]$ is an irreducible polynomial of degree 5 with exactly 3 real roots. Show that the Galois group of P is the symmetric group S_5.

(5) Let R be a commutative ring with 1, and let M be an R-module. Show that the following statements are equivalent:
(i) $M = 0$;
(ii) $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of R; and
(iii) $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of R.

(6) Let R be a commutative ring with 1.
(a) Establish that every R-module is projective if and only if every R-module is injective.
(b) Provide an example of an R-module that is projective, but not injective.
(c) Provide an example of an R-module that is injective, but not projective.