ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, WINTER 2024

Instructions:

- You should complete this exam in **six hours**. You are allowed to take a 20-minute break at your convenience, for a total of 6 hours and 20 minutes. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: January 10, 2024.

(1) Let G be a finite group. Let
$$C_G(g) := \{h \in G : hg = gh\}$$
be the centralizer of an element $g \in G$.

(a) Prove that $C_G(g)$ is a subgroup of G.

(b) Prove that the number of elements of G conjugate to g is $[G : C_G(g)]$.

(2) Let A and B denote two commutative rings with unit, and let $f \colon A \to B$ be a surjective ring homomorphism. Prove that if $I \subseteq A$ is an ideal, then $f(I)$ is an ideal of B. Show this statement can be false if f is not surjective.

(3) Are the 3×3 complex matrices
$$A = \begin{pmatrix} 5 & 6 & 0 \\ -3 & -4 & 0 \\ -2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & -1 & 2 \\ -10 & 6 & -14 \\ -6 & 3 & -7 \end{pmatrix}$$
similar? Are they diagonalizable?

(4) Let F be a field, and let $f(x) \in F[x]$ be a separable irreducible polynomial of degree n. Let L denote the splitting field of $f(x)$. For every prime p dividing n, show that $\mathrm{Gal}(L/F)$ has an element of order p. Give an example with $n = 4$ where the Galois group does not contain an element of order 4.

(5) Let A be a commutative ring with unit. For $f \in A$, let A_f be the localization of A with respect to multiplicative closed set $\{1, f, f^2, \ldots\}$. Prove that if $f, g \in A$ are elements such that $g - f$ is nilpotent, then A_f and A_g are isomorphic as A-algebras.

(6) A map of commutative rings $f \colon A \to B$ is *faithfully flat* if it is flat and $N \otimes_A B \neq 0$ for all nonzero A-module N. Prove that if A and B are local rings and f is a flat map such that $f^{-1}(m_B) = m_A$ then f is faithfully flat.