ALGEBRA QUALIFYING EXAMINATION

RICE UNIVERSITY, SPRING 2020

Instructions:

- You should complete this exam in a single **four** block of time. Attempt all **six** problems.
- The use of books, notes, calculators, or other aids is **not** permitted.
- Justify your answers in full, carefully state results you use, and include relevant computations where appropriate.
- Write and sign the Honor Code pledge at the end of your exam.

Date: May 11th, 2020.

(1) Let p be a prime number.

   (a) Classify groups of order p^2.

   (b) Prove that there exists a nonabelian group of order 75.

(2) Let M be a finite-dimensional $\mathbb{R}$-vector space, and let $T\colon M \to M$ be a linear map. Consider M as an $\mathbb{R}[t]$-module, where $t \cdot v := T(v)$ for $v \in M$. Suppose that M is generated, as an $\mathbb{R}[t]$-module, by two elements v_1 and v_2 that are subject to the relations

$$(t^2 - t - 2)v_1 + (-t^2 - 2t + 8)v_2 = 0,$$
$$(t^2 - 2t)v_1 + (-t^2 + 4)v_2 = 0.$$

   (a) Determine the dimension of M as an $\mathbb{R}$-vector space.

   (b) Compute the rational canonical form associated to T.

(3) Let $k[x]$ be a polynomial ring over a field k, and let $f(x) = x^4 + 8x + 12$.

   (a) Determine the Galois group of $f(x)$ when $k = \mathbb{Q}$.

   (b) Determine the Galois group of $f(x)$ when $k = \mathbb{F}_{17}$ (the finite field with 17 elements).

(4) Let k be a field, and consider $k[x]$ as a $k[y]$-module via the k-homomorphism

$$f\colon k[y] \to k[x],$$
$$y \mapsto x^2.$$

   Is the ring $k[x] \otimes_{k[y]} k[x]$ a domain? Justify your answer.

(5) Let A be a commutative ring with unit. Endow the prime spectrum $\operatorname{Spec}(A)$ with its usual Zariski topology, whose closed sets are of the form

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} A : I \subseteq \mathfrak{p}\},$$

   where $I \subseteq A$ is any ideal. Prove that $\operatorname{Spec}(A)$ is disconnected if and only if A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

(6) Let A be a commutative ring with unit, and let $S \subseteq A$ be a multiplicatively closed subset. Let M and N be A-modules. Prove that $S^{-1}(M \otimes_A N)$ and $S^{-1}M \otimes_{S^{-1}A} S^{-1}N$ are isomorphic as $S^{-1}A$-modules.