Analysis Exam, January 2021

Please put your name on your solutions, use 8 1/2×11 in. sheets, and number the pages.

1. For a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \neq 0$ for all n, we say that it is *subexponential* if

$$\limsup_{n\to\infty} \frac{1}{n} \log|x_n| \leq 0.$$

If the sequence $(x_n)_{n=1}^{\infty}$ is subexponential, prove that the sequence $y_n = \sup_{1\leq k\leq n}|x_k|$ is subexponential.

2. Let μ be a finite positive Borel measure on $\mathbb{C}$ such that $\mu(\mathbb{C} \setminus \overline{\mathbb{D}}) = 0$. Prove that the function

$$f(z) = \int \frac{1}{w-z} d\mu(w)$$

is well-defined on $\mathbb{C} \setminus \overline{\mathbb{D}}$ and that $\lim_{z\to\infty} zf(z) = -\mu(\overline{\mathbb{D}})$.

3. Let $\Omega \subset \mathbb{C}$ be a bounded domain whose boundary is a finite union of Jordan curves. Let $f(z)$ be holomorphic in the domain Ω with a continuous extension to $\Omega \cup \partial\Omega$. Suppose that on $\partial\Omega$, we have that either $|\operatorname{Re} f| = 2$ or $|\operatorname{Im} f| = 2$. Show that either f is constant or there is a point $z_0 \in \Omega$ so that $f(z_0) = 1$.

4. Find the value of the improper integral

$$\int_0^\infty \cos(t^2)dt = \lim_{L\to\infty} \int_0^L \cos(t^2)dt.$$

5. Fix $\alpha \in (0,1)$. The space $C^\alpha[0,1]$ is the space of continuous functions on $[0,1]$ with

$$\|f\|_{C^\alpha} = \sup|f| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty,$$

equipped with norm $\|\cdot\|_{C^\alpha}$.

a) Show that the unit ball of $C^\alpha[0,1]$ has compact closure in $C[0,1]$.

b) Show that $C^\alpha[0,1]$ is of first category in $C[0,1]$, i.e., that its complement is generic in the sense of Baire category.

6. Consider the differential equation $y'' = f(y)y$ on the domain $x \in [0,\infty)$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is strictly positive, monotone increasing, and continuous. Show there is a positive strictly decreasing solution $y(x)$ to this equation on the interval $[0,\infty)$ with $y(0) = 1$.