Analysis Exam, August 2022

Please put your name on your solutions, use different 8 1/2×11 in. sheets for different problems, and number the pages. In writing complete solutions, take care to give clear references to well-known results that you use.

1. Compute the integral
$$\int_{-\infty}^{\infty} \frac{x \sin 3x}{x^2 - 4x + 5} dx.$$

2. (a) Let $1 \leq p < \infty$. Show that if a sequence of real-valued functions $\{f_n\}$ converges in $L^p(\mathbb{R})$, then it contains a subsequence that converges almost everywhere.
   (b) Give an example of a sequence of functions converging to 0 in $L^2(\mathbb{R})$ that does not converge almost everywhere.

3. Suppose that f and g are holomorphic on the punctured unit disk $0 < |z| < 1$.
   (a) If
   $$\sup_{0<|z|<1} |z|^{1/3}|f(z)| < \infty,$$
   is the singularity 0 of f necessarily *removable*? (i.e. is f the restriction of a function holomorphic on the whole unit disk?).
   (b) If
   $$\sup_{0<|z|<1} |z|^{4/3}|g'(z)| < \infty,$$
   is the singularity 0 of g necessarily removable?

4. Prove that for every $f \in C([0,1])$,
$$\lim_{a\to\infty} a \int_0^1 e^{-ax} f(x)\, dx = f(0).$$
   Hint: compute the limit for $f(x) = x^k$, $k = 0, 1, 2, \ldots$

5. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Find the set of analytic bijections $f : \mathbb{D} \setminus \{0, 1/2\} \to \mathbb{D} \setminus \{0, 1/2\}$.

6. Let μ be a positive measure and let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L^1(d\mu)$. Prove that for all $\epsilon > 0$ there exists $\delta > 0$ such that for any measurable set E,
$$\mu(E) < \delta \implies \forall n \quad \left| \int_E f_n d\mu \right| < \epsilon.$$