Analysis Qualifying Exam, August 2021

Please put your name on your solutions, use a single 8 1/2×11 in. sheet for each problem, and number the pages. In writing complete solutions, take care to give clear references to well-known results that you use.

1. Let K be a compact metric space. Let $f_n : K \to \mathbb{C}$ be continuous functions for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ and let $f : K \to \mathbb{C}$ be a continuous function.

   Prove that the functions f_n converge to f uniformly on K as $n \to \infty$ if and only if the function $g : K \times (\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}) \to \mathbb{C}$ defined by

   $$g\left(x, \frac{1}{n}\right) = f_n(x), \qquad g(x, 0) = f(x)$$

   is a continuous function on $K \times (\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\})$.

2. For $-1 < a < 2$, compute the improper integral

   $$\int_0^{+\infty} \frac{x^a}{1+x^3}\, dx$$

3. Consider the vertical strip $\{z \in \mathbb{C} \mid \operatorname{Re} z \in (0,1)\}$ in the complex plane. Find all functions $f(z)$ which are holomorphic in the strip and continuous on the closed strip $\{z \in \mathbb{C} \mid \operatorname{Re} z \in [0,1]\}$, purely imaginary on the boundary edge $x = 0$, have $\operatorname{Re} f(z) = 7$ on the boundary edge $x = 1$, and which satisfy $|f(z)| \le A + |z|^n$ for some fixed positive integer n and some positive real number $A > 0$.

4. Let $p(z)$ be a polynomial. Suppose the zeroes of $p(z)$ lie in the right half plane $\{\operatorname{Re} z > 0\}$. Show that the zeroes of the derivative $p'(z)$ also lie inside right half plane.

   Hint: consider $\frac{p'(z)}{p(z)}$ *and write this out as a sum of functions with simple poles at the zeroes of* $p(z)$*. Rewrite each term as a fraction with a real denominator and then study the expression at a zero of* $p'(z)$*.*

5. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of positive measurable functions on a measure space $(X, \mathscr{B}, \mu)$. Suppose that the sum

   $$\sum_{k=1}^{\infty} \mu\{x \in X | f_k(x) \ge \epsilon\}$$

   converges for every $\epsilon > 0$. Prove that $f_k(x) \to 0$ almost everywhere on X.

6. Let $\{f_n\}$ be a sequence in $L^2([0,1])$ with uniformly bounded L^2 norm: $\|f_n\| < 17$. Show that if f_n converges to zero in measure (with respect to Lebesgue measure), then f_n converges to zero in L^1.

   Recall that "f_n converges to zero in measure" if, for every $\epsilon > 0$, *we have* $\lim_{n\to\infty} \mu(\{|f_n(x)| > \epsilon\} = 0$.