Analysis Qualifying Exam, May 2021

Please put your name on your solutions, use a single 8 1/2×11 in. sheet for each problem, and number the pages. In writing complete solutions, take care to give clear references to well-known results that you use.

1. (a) Consider a sequence of complex numbers such that $\operatorname{Re} z_j \geq 0$ for all j. If the series $\sum_{j=1}^{\infty} z_j$ and $\sum_{j=1}^{\infty} z_j^2$ are convergent, prove that $\sum_{j=1}^{\infty} |z_j|^2$ is convergent.

   (b) Prove that there exists a sequence z_j of complex numbers such that $\sum_{j=1}^{\infty} z_j$ and $\sum_{j=1}^{\infty} z_j^2$ are convergent, but $\sum_{j=1}^{\infty} |z_j|^2$ is divergent.

2. Let $f \in L^1([0,1])$. If
$$\int_0^x f(t)\, dt = 0 \qquad \forall x \in [0,1]$$
prove that $f = 0$ Lebesgue-a.e..

3. For $-1 < a < 2$, compute the improper integral
$$\int_0^{+\infty} \frac{x^a}{1+x^3}\, dx$$

4. Let $f_n : \mathbb{R} \to [0,1]$ be a sequence of measurable functions with $\sup_x f_n(x) = \frac{1}{n}$ but $\int f_n(x)dx = 1$ for all n. Let $F(x) = \sup_n f_n(x)$. Find, with proof, whether necessarily $\int F(x)dx = \infty$.

5. Prove that for all $f \in C^1([0,1])$,
$$\lim_{n\to\infty} \left(\int_0^1 (n+1)(n+2)x^n f(x)\, dx - (n+2)f(1) \right) = -f'(1).$$

6. Let U be a bounded domain in the plane that contains the origin. Let $f(x)$ be a holomorphic map of U with image in U. Suppose that the Taylor series of f at the origin is
$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$
Show $a_2 = 0$.

   Hint: consider iterates $f^{(n)}(z) = f \circ f \circ f \circ \dots . \circ f$ *(n times).*