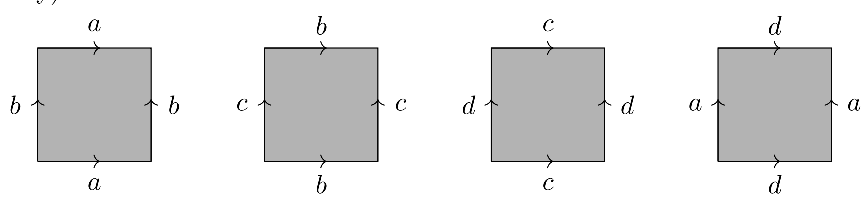
RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - JANUARY 2021

This is a 4 hour, closed book, closed notes exam. There are six problems; complete all of them. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam. Upload to Gradescope when you are finished.

Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

1. Let X be the connected 2–dimensional space obtained by gluing the sides of the four squares (homeomorphically) as shown.



   (a) Compute the homology groups $H_k(X)$, for all $k \geq 0$.
   (b) Compute a presentation for $\pi_1(X)$.
   (c) Prove that for any two generator group G, there is a connected regular covering space $p\colon \tilde{X} \to X$ for which the covering group is G.

2. Let W be the wedge product of two copies of $\mathbb{RP}^2$. Prove that any map from W to the circle S^1 lifts to the universal cover of S^1, $W \to \tilde{S^1}$

3. (a) State the Mayer-Vietoris theorem
   (b) Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0,1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. That is,
$$S(X) := X \times [0,1]/\sim$$
   where $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for all $x, y \in X$. Describe the relation between the homology groups of X and $S(X)$.

4. Let M, N be closed (compact and no boundary), connected, orientable 4-dimensional manifolds with $\pi_1(M) = 1$ and $H_p(M) \cong H_p(N)$ for all $p \geq 2$. Prove or disprove: M and N are homotopy equivalent.

5. Let X and Y be closed, connected, oriented n-dimensional manifolds without boundary and let $f : X \to Y$ be a continuous map. Suppose $\beta_p(X) < \beta_p(Y)$ for some $p > 0$.
   (a) Prove that $f^* : H^p(Y;\mathbb{Q}) \to H^p(X;\mathbb{Q})$ has a non-trivial kernel.
   (b) Show that f is a degree zero map.

6. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere defined by $x^2 + y^2 + z^2 = 1$ and $Z \subset \mathbb{R}^3$ be the subset defined by the equation $x^3 + 3xy^2 + 3xz^2 = 1$. Prove that $Z \cap S^2$ is a 1–dimensional submanifold of S^2.