RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - JANUARY 2024

This is a closed book, closed notes exam. **Justify all of your work** as much as time allows. **Only turn in solutions to 6 of the 9 problems**. You will have 6 hours to complete the exam. Write and sign the Rice honor pledge at the end of the exam:

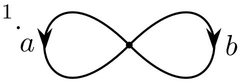
Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

Notation/Terminology: By "map" we mean "continuous map". A *closed manifold* is compact manifold without boundary.

1. Suppose $f: Z \to Z$ is a self-map of a path connected space Z, and M_f is the mapping torus of f. Construct a map to the circle $\varphi: M_f \to S^1$ such that $\varphi_*: \pi_1(M_f) \to \pi_1(S^1)$ is surjective. *Recall: M_f is the quotient space of $Z \times [0,1]$ identifying $(z,0)$ to $(f(z),1)$, for all $z \in Z$.*

2. Suppose $f: \Sigma_2 \to \Sigma_3$ is a map from a closed, orientable surface of genus 2 to a closed, orientable surface of genus 3. Prove that $f_*(\pi_1(\Sigma_2))$ has infinite index in $\pi_1(\Sigma_3)$.

3. Let X be the space obtained by attaching 2 disks to the wedge of circles as shown below, so that the boundaries of the disks trace out the loops $abab^{-1}$ and $ab^2a^{-1}b^{-1}$.



   (a) Prove there is a unique connected, 2-sheeted covering space $\widetilde{X} \to X$, up to isomorphism.

   (b) Compute the homology groups of the covering space from part (a), $H_p(\widetilde{X})$ for all $p \geq 0$.

4. Let W be the quotient space of the solid torus $S^1 \times D^2$ obtained by identifying all points in the boundary $S^1 \times \partial D^2$ to a point (that is, $W = (S^1 \times D^2)/(S^1 \times \partial D^2)$). Compute the reduced homology groups $\widetilde{H}_p(W)$, for all $p \geq 0$.

5. Let N be a closed, connected 5–manifold with $\pi_1(N) \cong \mathbb{Z}_{15}$ and $H_2(N) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$.

   (a) Prove that N is orientable.

   (b) Compute $H_p(N)$ and $H^p(N;\mathbb{Z})$, for all $p \geq 0$.

6. Prove that there is no degree 1 map $f: S^2 \times S^2 \to \mathbb{CP}^2$.

7. (a) Prove that $F([x:y:z]) = [x^2+y^2 : y^2+z^2]$ well-defines a smooth map $F: \mathbb{RP}^2 \to \mathbb{RP}^1$.

   (b) Prove that $F^{-1}([1:2])$ is a smooth submanifold, and determine its dimension.

8. (a) Prove that if θ is a smooth $(n-1)$–form on a closed, smooth n–manifold, then $d\theta$ is zero at some point.

   (b) Find an example of a smooth $(n-1)$–form θ on a compact, smooth n–manifold-with-boundary for which $d\theta$ is nowhere zero.

9. Let Δ be the distribution on $\mathbb{R}^3$ defined by the form $\omega = dz - y\,dx$; that is, ω generates the ideal of forms vanishing on Δ.

   (a) Use ω to decide whether or not Δ is integrable.

   (b) Find vector fields generating Δ and use them to give an alternate proof of (a).