RICE UNIVERSITY TOPOLOGY QUALIFYING EXAM - MAY 2021

This is a 4 hour, closed book, closed notes exam. There are six problems; complete all of them. Justify all of your work, as much as time allows. Write and sign the Rice honor pledge at the end of the exam. Upload to Gradescope when you are finished.

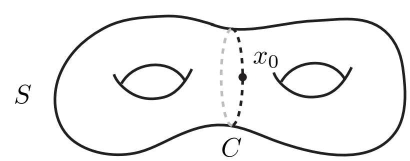
Honor pledge: On my honor, I have neither given nor received any unauthorized aid on this exam.

You should assume all your spaces are path-connected unless otherwise stated. In this exam, $\mathbb{Z}_n$ means the finite cyclic group of order n.

1. Let X be the space obtained by gluing a disk to "the waist curve" C in a genus 2 surface S as shown. More precisely, let $f\colon \partial D^2 \to C$ be a homeomorphism, and define

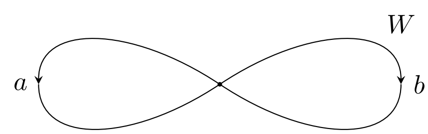
$$X = S \sqcup D^2/\sim$$

where $x \sim f(x)$ for all $x \in \partial D^2$.



   (a) Compute the homology groups $H_n(X)$ of the space X for all n.
   (b) Compute a presentation for $\pi_1(X, x_0)$, based at a point $x_0 \in C$.
   (c) Prove that there is a homomorphism $\pi_1(X, x_0) \to F_2$, onto the free group F_2 of rank 2.

2. Let W be a wedge of circles shown to the right. For each cover you are to construct below, draw the graph and label the edges.



   (a) Construct a connected regular cover $p_1\colon \tilde{W}_1 \to W$ of degree 3.
   (b) Construct a connected *irregular* cover $p_2\colon \tilde{W}_2 \to W$ of degree 3.
   (c) Construct a connected regular cover $p_3\colon \tilde{W}_3 \to W$ with non-abelian covering group.
   (d) Prove that there is no connected regular cover $p\colon \tilde{W} \to W$ with covering group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

3. Suppose $X = \mathbb{RP}^2$ and $Y = \mathbb{R}^2/\mathbb{Z}^2$, and let $\tilde{X}$ and $\tilde{Y}$ be the respective universal coverings.
   (a) What familiar spaces are $\tilde{X}$ and $\tilde{Y}$?
   (b) Prove that any map $f\colon X \to Y$ admits a lift $\tilde{f}\colon X \to \tilde{Y}$.
   (c) Construct a map $g\colon Y \to X$ that does *not* admit a lift $\tilde{g}\colon Y \not\to \tilde{X}$.

4. Let X be a closed (compact with boundaryless), connected 5-dimensional manifold such that $\pi_1(X) \cong \mathbb{Z}_{25}$ and $H_2(X) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_7$.
   (a) Prove that X and all of its covering spaces are orientable.
   (b) Prove that $\chi(X) = 0$.
   (c) Calculate $H_p(X; \mathbb{Z})$ for each p.

5. Let X be a closed (compact and boundaryless), oriented 4-manifold with $\beta_2(X) \neq 0$. Prove that any continuous map $f : S^4 \to X$ has degree equal to 0.

6. Consider the subspace $M \subset \mathbb{RP}^3$ defined by the equation
$$x^3 + y^3 + z^3 + w^3 = 0$$
in $\mathbb{R}^4$. Prove that M is a smooth, codimension 1 submanifold.