THE SECOND MIDTERM
REPRESENTATION THEORY 374.

2 problems for a C; 5 problems for a B; 8 problems for an A.

Each problem with asterisks counts as two.

In all problems, please give detailed solutions.

Please clearly formulate all theorems that you are using.

Consultation with other people is not allowed.

You may use any materials.

No credit for wrong answers.

Please write the Honour Pledge in full.

This midterm is due on Thursday 7 April in class.

1 Lie Groups Over Finite Fields.

Let $p$ be a prime number, and let $\mathbb{Z}/p\mathbb{Z}$ be the field with $p$ elements. Define

$$SL_2(\mathbb{Z}/p\mathbb{Z}) = \left\{ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} : x_{ij} \in \mathbb{Z}/p\mathbb{Z}, x_{11}x_{22} - x_{12}x_{21} = 1 \right\}.$$

Let $D_2(\mathbb{Z}/p\mathbb{Z})$ be the subgroup of diagonal matrices such that $x_{11} = x_{22}$.

1. Show that $D_2(\mathbb{Z}/p\mathbb{Z})$ is a normal subgroup of $SL_2(\mathbb{Z}/p\mathbb{Z})$. How many elements does it have for $p = 2$? For $p = 3$?

2. Define $PSL_2(\mathbb{Z}/p\mathbb{Z}) = SL_2(\mathbb{Z}/p\mathbb{Z})/D_2(\mathbb{Z}/p\mathbb{Z})$. Find the number of elements of $PSL_2(\mathbb{Z}/p\mathbb{Z})$ for $p = 2$ and $p = 3$.

We now introduce the projective line $P(\mathbb{Z}/p\mathbb{Z})$. It is defined in the usual way: two nonzero vectors $(x_1, x_2)$ and $(x'_1, x'_2)$ in $(\mathbb{Z}/p\mathbb{Z})^2$ are considered equivalent if there exists a $y \in \mathbb{Z}/p\mathbb{Z}$ such that $x_i = yx'_i$ for each $i$. The quotient of $(\mathbb{Z}/p\mathbb{Z})^2 \setminus 0$ by this equivalence relation is called the projective line over $\mathbb{Z}/p\mathbb{Z}$, and is denoted $P(\mathbb{Z}/p\mathbb{Z})$.

3. Find the number of points in $P(\mathbb{Z}/p\mathbb{Z})$ for $p = 2$ and $p = 3$.

4. **. Consider the groups $PSL_2(\mathbb{Z}/2\mathbb{Z})$ and $PSL_2(\mathbb{Z}/3\mathbb{Z})$. One of them is isomorphic to a symmetric group $S_k$ for a certain value of $k$, while the other is isomorphic to the alternating group $A_l$ for a certain value of $l$. Find $k$ and $l$, and establish the isomorphisms. Hint: study the action on the projective line.

2 Sums of Four Squares.

5. Consider the equation

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2.$$

Find the $z_i$ as polynomials in $x_1, \ldots, x_4$ and $y_1, \ldots, y_4$ with integer coefficients, such that the above equation holds.
3 Homomorphisms of Lie Groups.

6. Consider the homomorphism $SL_2 \mathbb{R} \rightarrow SO_{2,1} \mathbb{R}$ constructed in class. Find its kernel.

7. Do the same for $SL_2 \mathbb{C} \rightarrow SO_3 \mathbb{C}$.

8. Do the same for $SU_2 \times SU_2 \rightarrow SO_4 \mathbb{R}$.

4 Exponentials of Matrices.

9. Give an explicit example of two matrices $A$ and $B$ such that $e^{A+B} \neq e^A e^B$.

10. Let $A, B$ be elements of $\text{Mat}_n(\mathbb{C})$. Find

$$\frac{d^2}{dt^2} e^{tA} e^{tB} e^{-tA} e^{-tB} \bigg|_{t=0}.$$