PROBABILITY THEORY

THE FIRST MIDTERM

9 PROBLEMS

3 problems for a C; 5 problems for a B; 7 problems for an A.

This midterm is due on Tuesday 17 Mar. in class.

In all problems, please give detailed solutions.

Please clearly formulate all theorems that you are using.

You may use the textbook and your notes, but no other materials.

Consultation with other people is not allowed.

Please write the Honour Pledge in full.

No partial credit for wrong answers.

1 The Polya Urn Scheme

Feller, Chapter V.8, p. 131.

1. Problems 18,19.
2. Problems 20,21.
3. Problems 22,23.

Remark. The notation means: \((x)_n = x(x-1) \cdots (x-n+1)\).

2 Upper and Lower Limits of Sets.

For a sequence of sets \(A_n\), its upper limit \(\limsup\) \(\lim_{n \to \infty} A_n\) is defined by the formula

\[
\limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k,
\]

its lower limit \(\liminf\) \(\lim_{n \to \infty} A_n\) by the formula

\[
\liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.
\]

and the sequence \(A_n\) is said to have a limit \(A_{\infty} = \lim_{n \to \infty} A_n\) if \(\limsup A_n = \liminf A_n = A_{\infty}\).
5. Show that $\limsup_{n \to \infty} A_n$ consists of elements that belong to infinitely many $A_n$; that $\liminf_{n \to \infty} A_n$ consists of elements that belong to all $A_n$ except finitely many. Show $\liminf_{n \to \infty} A_n \subset \limsup_{n \to \infty} A_n$, and give examples when these two sets coincide and when they don’t. Derive formulas for the upper and the lower limits of complements.

6. For a sequence of sets $A_n$, compare the sets $\limsup(A_n \setminus A_{n+1})$ and $\limsup(A_{n+1} \setminus A_n)$. **Remark.** Examples of possible answers: the first always contains the second, or vice versa, or these sets need not be comparable by inclusion.

7. Let $A_n$ be measurable with respect to a probability measure $P$. Show that if $A_\infty = \lim_{n \to \infty} A_n$, then $P(A_\infty) = \lim_{n \to \infty} P(A_n)$.

3 The Hahn-Vitali-Saks Theorem.

8. Let $\Omega$ be a set, $\mathcal{F}$ a sigma-algebra on $\Omega$. Let $P_n$ be a sequence of probability measures on $\mathcal{F}$. Assume that for each $A \in \mathcal{F}$ there exists the limit $\lim_{n \to \infty} P_n(A) = p(A)$. Show that $p(A)$ is a probability measure on $\mathcal{F}$.

9. Under the assumptions of the previous problem, if $A_k$ is a sequence of sets satisfying $A_{k+1} \subset A_k$, $\bigcap_k A_k = \emptyset$, then $\lim_{k \to \infty} \left( \sup_n P_n(A_k) \right) = 0$. 