Final Exam
Math 212 Fall 2010

Instructions: This is a 180 minute exam. You should work alone, without access to any book or notes. No calculators are allowed. Do not discuss this exam with anyone other than your instructor or TA. When you have completed the exam, write out and sign the Honor Code pledge on the front.

The exam consists of 11 questions. You must show all of your work on each problem to receive full credit, and be sure to clearly indicate your final answer to each question. Points may be deducted for incorrect, irrelevant or incoherent statements.

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Write out the Honor Pledge:

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\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta
\]

\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]

\[
\sin^2 \theta = \frac{1 - \cos 2\theta}{2}
\]

\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}
\]
1. [20 Points] For this problem only, you need not write any reasons or show any work. Mark each of the following quantities as one of

CS constant scalar or constant scalar-valued function,
CV constant vector or constant vector-valued function,
SF scalar function: could be non-constant, depending on what \( g, F \), etc. are,
VF vector field: could be non-constant, depending on what \( g, F \), etc. are, or
ND not defined.

- \( c \) is a real number
- \( u \) is a fixed unit vector in \( \mathbb{R}^3 \)
- \( v \) is a fixed vector in \( \mathbb{R}^3 \)
- \( g(x, y, z) \) is a fixed but arbitrary function \( \mathbb{R}^3 \to \mathbb{R} \)
- \( F(x, y, z) \) is a fixed but arbitrary vector field \( \mathbb{R}^3 \to \mathbb{R}^3 \)
- \( C \) is curve from (0, 0, 0) to (2, -5, 3)
- \( S \) is the surface of the unit sphere, centered at the origin and oriented outward
- \( W \) is the region inside the unit sphere \( S \)

\[
\begin{align*}
\text{CS} & \quad \text{CV} & \quad \text{SF} & \quad \text{VF} & \quad \text{ND} \\
\quad & u \cdot (cv) & u \times F + F \times u & \text{curl } F & \text{div } g \\
\quad & \text{div } g & \text{curl } \nabla g & \nabla g & \frac{\partial}{\partial y} \|F\|^2 \\
\quad & gF + c & \nabla (\text{div } F) & \text{div } (F \times F) & u \times v - u \cdot v \\
\quad & \text{div } (\text{curl } F) & \nabla (v \cdot F) & \text{curl } F + (\text{div } F)u & \|\text{curl } F\| \\
\quad & \text{the directional derivative of } g \text{ in the direction } u \\
\quad & \int \int \int_W (\text{div } F) \, dV & \int \int_S \|F\| \, dS & \int \int_C \nabla g \, ds
\end{align*}
\]

Note that the answer for \( F - 3F + 2F \) would be \( \text{CV} \), rather than \( \text{VF} \), since \( F - 3F + 2F = 0 \) is a constant vector field no matter what the vector field \( F \) is.

Assume that \( g \) and \( F \) are twice continuously differentiable.
2. [15 Points] Find the maximum and minimum values (or explain why they do not exist) of the function \( f(x, y) = e^{2xy} \), subject to the constraint \( x^3 + y^3 = 16 \).
3. [20 Points] Compute \( \iint_T x \, dS \), where \( T \) is the triangle with vertices (1, 1, 1), (2, 0, 3), and (2, 1, 1).
4. [20 Points]

(a) Find a vector field \( \mathbf{F}(x, y) = (P(x, y), Q(x, y)) \) on \( \mathbb{R}^2 \) so that

\[
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1
\]

on all of \( \mathbb{R}^2 \).

(b) Using Green’s theorem, compute the area enclosed by the parametric curve \( \mathbf{c}(t) = (t - t^3, t - t^2) \), for \( 0 \leq t \leq 1 \).
5. [15 Points] Evaluate the following limits or show that they do not exist.

(a) \( \lim_{(x,y) \to (0,0)} \frac{1 - \cos(xy^2)}{x^2y^4} \).

(b) \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^3}{x^2 + y^2} \).
6. [20 Points] Use the divergence theorem to evaluate the flux integral

\[ \iint_S (xz, x^2y, y^2z) \cdot d\mathbf{S}, \]

where \( S \) is the whole surface of the solid cylinder \( x^2 + y^2 \leq 1, \ 0 \leq z \leq 3 \), oriented outwards.
7. [15 Points]

(a) Compute the Hessian matrix $Hf$ of the function $f(x, y, z) = x^2 + 2xy + 4xz + 3y^2 + z^2$.

(b) Use the second derivative test to determine whether the critical point $(0, 0, 0)$ of $f$ is a local maximum, local minimum, saddle type critical point, or degenerate critical point.
8. [15 Points] Let $D$ be the region in the plane defined by the inequalities $x^2 + y^2 \leq 9$ and $x^2 + (y - 1)^2 \geq 1$.

   (a) Sketch the region $D$.

   (b) Let $f(x, y) = \sqrt{x^2 + (y - 1)^2} - 1$. Write $\iint_D f\,dA$ as a sum or difference of iterated integrals with respect to the variables $x$ and $y$. (Your integrals may be in the “$dx\,dy$” order or the “$dy\,dx$” order or a mix of the two.) \textbf{You need not evaluate the integrals.}
(a) Suppose that quantities $x, y, z, u, v, a$ are related to one another by the equations

$$a = f(x, u, v), \quad u = g(x, v), \quad \text{and} \quad v = h(x, y, z).$$

Then $a$ is a function of $x, y, \text{and} \ z$. Compute $\frac{\partial a}{\partial x}$ in terms of the partial derivatives of $f, g, \text{and} \ h$.

(b) Let $F(x, y, z) = x^3 + 2xyz + y^3 - z^3$. Then, in a neighborhood of the point $(2, 1, -3)$, the equation $F(x, y, z) = 24$ determines $y$ as a function of $x$ and $z$, i.e. there is a function $y = f(x, z)$ so that $F(x, f(x, z), z) = 24$ for all $(x, z)$ near $(2, -3)$. Compute $\frac{\partial f}{\partial z}(2, -3)$.
10. [20 Points] Let \( \mathbf{F}(x, y, z) = \left( 2xyz e^{x^2} \right) \mathbf{i} + \left( \sqrt{1 + y^2} \right) \mathbf{j} + \left( x^2 e^{x^2 z} + z^2 \right) \mathbf{k} \).

(a) Compute curl \( \mathbf{F} \).

(b) Let \( \mathbf{c}(t) = (\cos t \sin t, \sin^2 t, t) \), for \( 0 \leq t \leq \pi \). Evaluate the line integral \( \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \).
11. [20 Points] Let $S$ be the part of the sphere $x^2 + y^2 + z^2 = 9$ between the plane $z = 0$ and the cone $z = \sqrt{x^2 + y^2}$.

(a) Describe the surface $S$ in spherical coordinates (using equations and/or inequalities).

(b) Evaluate $\iint_S (\text{curl } F) \cdot dS$, where $S$ is oriented outwards and $F(x, y, z) = \left(y - z, \frac{x}{2}, e^{z-x^2y}\right)$.