1. Do exercise 15 and major exercise 6 from chapter 2 in the textbook.

Betweenness

2. Find (and explain) the flaw in the false “proof” that all triangles are isosceles on pages 25-27 in the textbook.

3. Do exercises 1, 8, and 15 from chapter 3 in the textbook. [Exercises 1 and 8 have been moved to the next homework.]

Combinatorics of plane tilings

4. Consider the standard tiling of the Euclidean plane by regular hexagons. Given two hexagons \( H \) and \( H' \) of the tiling, we can define the (combinatorial) distance between them to be the smallest number \( n \) so that there exist hexagons of the tiling \( H = H_0, H_1, \ldots, H_n = H' \) so that for each \( i = 1, \ldots, n \), the hexagons \( H_{i-1} \) and \( H_i \) share an edge. (See below figure.)

Fix a hexagon \( H \) of the tiling, and let \( a_n \) be the number of hexagons of the tiling whose distance from \( H \) is \( n \). Find (with proof\(^1\)) a formula for \( a_n \)\(^2\).

5. It turns out it is possible to tile the hyperbolic plane with regular heptagons, meeting three to a vertex\(^3\). As in the previous problem, fix a heptagon, and let \( b_n \) be the number of heptagons at distance \( n \) from it. Compute \( b_1, b_2, \) and \( b_3 \) and show that in general, \( b_{n+1} \geq 2b_n \)\(^4\).

\(^1\)Your proof should use the following axiom of induction, one of the standard axioms for the natural numbers \( \mathbb{N} \):

Suppose \( S \subseteq \mathbb{N} \) is a subset of the natural numbers such that

- \( 1 \in S \), and
- for all natural numbers \( n \), if \( n \in S \) then \( n + 1 \in S \).

Then \( S = \mathbb{N} \).

Thus if we want to prove that our formula for \( a_n \) holds for all natural numbers \( n \), we can consider the set \( S \subseteq \mathbb{N} \) consisting of all \( n \) for which the formula is true. To show \( S = \mathbb{N} \), we need only show that \( 1 \in S \), i.e. that the formula holds for \( n = 1 \), and that \( n \in S \Rightarrow (n + 1) \in S \), i.e. we must show that the formula holds for \( n + 1 \), assuming that we already know the formula holds for \( n \).

\(^2\)You can think of this as a discrete hexagonal version of finding the “circumference of a circle.”

\(^3\)The heptagons in question are all congruent to one another, although they don’t look like it in the Poincaré disk model for the hyperbolic plane as depicted below.

\(^4\)In particular, \( b_n \) grows exponentially, rather than linearly, as a function of \( n \). Later in the course, we’ll show analogously that the circumference of a circle in the hyperbolic plane grows exponentially as a function of its radius (see page 496 in the text).
6. Imagine a tiling of a “plane” by regular pentagons, meeting three to a vertex. As in the previous problems, fix a pentagon, and let $c_n$ be the number of pentagons at distance $n$ from it. Compute $c_n$ for $n \leq 4$.

Does your answer make sense? Could there really be a tiling of a “plane” by regular pentagons meeting three at a vertex? How about at tiling of a “plane” by squares meeting three at a vertex?

**Extra credit**

7. (a) Find (with proof) an explicit formula for the $b_n$ of problem 5.

[Hint: for $n \geq 2$, there are two different combinatorial “types” of heptagons at distance $n$ from the chosen one. Set $b_n = x_n + y_n$ based on the two types, find a recurrence for $x_n$ and $y_n$ that can be written in the form

$$
\begin{bmatrix}
x_{n+1} \\
y_{n+1}
\end{bmatrix} =
\begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix}
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix},
$$

and diagonalize the matrix $M$.]

(b) More generally, for a tiling by regular $p$-gons, meeting three at a vertex, find a formula for the number of $p$-gons at distance $n$ from a fixed one. [What happens differently when $p = 6$?]

(c) What does your formula say in the cases $p = 3, 4, 5$? Explain.