Instructions: You have two hours to complete this exam. You should work alone, without access to the textbook or class notes. You may not use a calculator. Do not discuss this exam with anyone except your instructor.

This exam consists of 6 questions. You must show your work to receive full credit. Be sure to indicate your final answer clearly for each question. Pledge your exam when finished, and include your name and section number on the front of the exam. The exam is due by Friday, 5 p.m. Good luck!

1. Let $l_1$ and $l_2$ be the intersecting lines given by the parameterizations

\[ l_1(t) = (1, 0, 1) + t(0, 2, 1), \]
\[ l_2(s) = (3, 3, 4) + s(-2, 1, -1). \]

(a) Find the angle between $l_1$ and $l_2$.
(b) Find a vector perpendicular to both $l_1$ and $l_2$.

Solution 1.

(a). [8 points] Let $v_1$ be the direction vector for the line $l_1$, and let $v_2$ be the direction vector for the line $l_2$. The angle $\theta$ between lines $l_1$ and $l_2$ satisfies

\[ v_1 \cdot v_2 = |v_1||v_2| \cos \theta \]

So

\[ (0)(-2) + (2)(1) + (1)(-1) = \sqrt{2^2 + 2^2 + 1^2} \sqrt{(-2)^2 + 1^2 + (-1)^2} \cos \theta \]
\[ \cos \theta = \frac{1}{\sqrt{30}} \]
\[ \theta = \arccos \frac{1}{\sqrt{30}} \]

(b). [7 points] Such a vector is $v_1 \times v_2$:

\[ v_1 \times v_2 = \begin{vmatrix}
  i & j & k \\
  0 & 2 & 1 \\
  -2 & 1 & -1 \\
\end{vmatrix} = -3i + 2j + 4k \]

Any scalar multiple of $(-3, 2, 4)$ is acceptable.

2. Let $f(x, y) = 4x^2 + y^2$.
(a) Sketch the level curves of $f(x, y) = k$ for $k = 2, 4, 8$.
(b) On your graph from part (a), draw a vector at the point $(1, 2)$ which gives the direction of the gradient of $f$ at that point.
(c) Graph and describe the $x = 0, y = 0$, and $z = 0$ cross sections of the graph of $f(x, y)$. Each cross section should appear on a separate graph.
(d) Sketch the graph of $f$.


3. Suppose a particle is travelling in $\mathbb{R}^2$ such that at time $t$ its position is given by 

$$c(t) = (t^2 - \pi t, \sin(t)).$$

(a) Find the velocity of the particle at time $t$. At what time(s) $t$ does the particle come to a stop?
(b) Give a parametric equation for the tangent line to $c(t)$ at time $t = \pi/4$.

Solution 3.

(a). [8 points] The velocity is $c'(t) = (2t - \pi, \cos t)$. We see $c'(t) = 0$ precisely at $t = \pi/2$.
(b). [7 points] Note that $c(\pi/4) = \left(\frac{-3\pi^2}{16}, \frac{1}{\sqrt{2}}\right)$ and $c'(t) = \left(\frac{-\pi}{2}, \frac{1}{\sqrt{2}}\right)$. The equation of the tangent line at $t = \pi/4$ is

$$\left(\frac{-3\pi^2}{16}, \frac{1}{\sqrt{2}}\right) + (t - \pi/4) \left(\frac{-\pi}{2}, \frac{1}{\sqrt{2}}\right)$$
4. The temperature at a point \((x, y)\) on a flat metal plate is given by the function

\[ T(x, y) = \frac{60}{1 + 2x^2 + y^2}, \]

where \(T\) is measured in °C and \(x, y\) are measured in meters. Find the rate of change of temperature with respect to distance at the point \((2, 1)\) in

(a) the \(y\)-direction,
(b) the direction given by the vector \(\mathbf{i} - 2\mathbf{j}\),
(c) the direction of the maximum rate of change.
(d) Is there a direction at \((2, 1)\) along which \(T\) does not change? If so, find a vector pointing in that direction.

**Solution 4.** Note that

\[ \nabla T(x, y) = \left( \frac{-240x}{(1 + 2x^2 + y^2)^2}, \frac{-120y}{(1 + 2x^2 + y^2)^2} \right) \]

(a). [5 points] The derivative in the \(y\)-direction at \((2, 1)\) is

\[ \nabla T|_{(2,1)} \cdot (0, 1) = \frac{-120}{(1 + 2 \cdot 2^2 + 1^2)^2} = \frac{-6}{5} \text{ °C/m} \]

(b). [5 points] First, we must normalize \((1, -2)\) to \(\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)\). The directional derivative at \((2, 1)\) is

\[ \nabla T|_{(2,1)} \cdot \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) = \left( \frac{-24}{5} \right) \left( \frac{1}{\sqrt{5}} \right) + \left( \frac{-6}{5} \right) \left( \frac{-2}{\sqrt{5}} \right) = \frac{-12\sqrt{5}}{25} \text{ °C/m} \]

(c). [5 points] The direction of maximum rate of change is \(\nabla T|_{(2,1)}\). First, normalize this to \(\frac{\nabla T|_{(2,1)}}{||\nabla T|_{(2,1)}||}\). Thus, the rate of change in this direction is

\[ \nabla T|_{(2,1)} \cdot \frac{\nabla T|_{(2,1)}}{||\nabla T|_{(2,1)}||} = \frac{||\nabla T|_{(2,1)}||^2}{||\nabla T|_{(2,1)}||} = ||\nabla T|_{(2,1)}|| = \sqrt{\left( \frac{-24}{5} \right)^2 + \left( \frac{-6}{5} \right)^2} = \frac{\sqrt{612}}{5} = \frac{6\sqrt{17}}{5} \]

(d). [5 points] We must find \(\vec{v} = (v_1, v_2)\) such that \(-\frac{24}{5}v_1 + \frac{-6}{5}v_2 = 0\). \(\vec{v} = (1, -4)\) works.

5. Let \(f(x, y, z) = x^3y^2z\), and let \(c(t) = (e^t, \sin(t), g(t))\), where \(g(t)\) is a differentiable function. Use a Chain Rule from vector calculus to find

\[ \frac{d}{dt} f(c(t)). \]

Your final answer will be in terms of \(g(t), g'(t)\) and \(t\).
Solution 5. [10 points] The chain rule says that \( \frac{df}{dt}(c(t)) = Df(c(t))Dc(t) \). Since
\[ Df(x, y, z) = (3x^2y^2z, 2x^3yz, x^3y^2) \]
and
\[ Dc(t) = \begin{pmatrix} e^t \\ \cos t \\ g'(t) \end{pmatrix} \]
we have
\[ \frac{d}{dt}f(c(t)) = \begin{pmatrix} 3e^{2t}\sin^2(t)g(t), 2e^{3t}\sin(t)g(t), e^{3t}\sin^2(t) \end{pmatrix} \begin{pmatrix} e^t \\ \cos t \\ g'(t) \end{pmatrix} \]
\[ = 3e^{3t}\sin^2(t)g(t) + 2e^{3t}\sin(t)\cos(t)g(t) + e^{3t}\sin^2(t)g'(t) \]
You must remember to evaluate \( Df \) at \( c(t) \).

6. Let
\[ f(x, y) = x^3 + 3x^2y^2 + y^3. \]
(a) Suppose a particle is sitting on the graph of \( f \) at the point \((-1, 1)\). In which direction should the particle move in order for its height to decrease most rapidly?
(b) Give the equation for the tangent plane of \( f(x, y) \) at the point \((-1, 1)\).
(c) Give the quadratic Taylor polynomial for \( f(x, y) \) at the point \((-1, 1)\).

Solution 6.

(a). [7 points] In order to find the direction for the height to decrease most rapidly, we need to find \(-\nabla f\), which is \(-\left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)\), and then evaluate at \((-1, 1)\). We have
\[ \frac{\partial f}{\partial x} = 3x^2 + 6xy^2 \]
\[ \frac{\partial f}{\partial y} = 6x^2y + 3y^2 \]
So \(-\nabla f\big|_{(-1,1)} = (3, -9)\).

(b). [6 points] A normal vector for the tangent plane is \((3, -9, 1)\), so the tangent plane is given by
\[ 3(x + 1) - 9(y - 1) + (z - 3) = 0 \]

(c). [7 points] Take the second order derivatives and evaluate at \((-1, 1)\).
\[ \frac{\partial^2 f}{\partial x^2}\big|_{(-1,1)} = 6x + 6y^2 = 0 \]
\[ \frac{\partial^2 f}{\partial y^2}\big|_{(-1,1)} = 6x^2 + 6y = 12 \]
\[ \frac{\partial^2 f}{\partial x\partial y}\big|_{(-1,1)} = 12xy = -12 \]
So the Taylor expansion is
\[ T(f(x, y)) = 3 + (-3)(x + 1) + 3(y - 1) + 6(y - 1)^2 + (-12)(x + 1)(y - 1) \]