1. The family 
\[ A = \{ A \subset \mathbb{N} : \text{either } A \text{ is finite or } \mathbb{N} \setminus A \text{ is finite} \} \]
is countable. To see this, note that 
\[ A = \bigcup_{k=0}^{\infty} A_k \cup B_k \]
where 
\[ A_k = \{ A \subset \{1, 2, \cdots, k\} \} \]
has a finite number \((2^k)\) of elements and 
\[ B_k = \{ \mathbb{R}^n \setminus A : A \in A_k \} \]
also has only a finite number of elements. A countable union of finite sets is countable.

2. There does exist a compact subset \( K \) of irrational numbers with positive 1-dimensional Lebesgue measure. One can take \( K = [0, 4] \setminus \bigcup_{i=1}^{\infty} (r_i - 2^{-i}, r_i + 2^{-i}) \) where 
\[ \{r_1, r_2, \cdots \} = \mathbb{Q}, \]
the rational numbers. Here \( \lambda(K) \geq 4 - \sum_{i=1}^{\infty} 2^{-i+1} = 2. \)

3. (a) For any subset \( A \) of \( \mathbb{R}^n \), \( \lambda^*(A) = \lambda(\cap_{i=1}^{\infty} G_i) \) for some open sets \( G_1, G_2, \cdots \) containing \( A \). In case \( \lambda^*(A) = \infty \), simply take \( G_i = \mathbb{R}^n \). For \( \lambda^*(A) < \infty \), choose open \( H_j \supset A \) with \( \lambda(H_j) < \lambda^*(A) + 1/j \). Letting \( G_i = \cap_{j=1}^{i} H_j \) we see that \( G_1 \supset G_2 \supset \cdots \), \( \lambda(G_1) < \infty \), and 
\[ \lambda^*(A) \leq \lambda(\cap_{i=1}^{\infty} G_i) = \lim_{i \to \infty} \lambda(G_i) \leq \lim_{i \to \infty} \lambda(H_i) \leq \lambda^*(A). \]
(b) Suppose \( Z = \cap_{i=1}^{\infty} G_i \setminus A \). If \( \lambda^*(Z) = 0 \), then \( Z \) is measurable and hence the difference \( A = (\cap_{i=1}^{\infty} G_i) \setminus Z \) is also measurable. Conversely, if \( A \) is measurable, then 
\[ \lambda(Z) = \lambda(\cap_{i=1}^{\infty} G_i) - \lambda(A) = 0 \]
because \( \lambda(\cap_{i=1}^{\infty} G_i) = \lambda(A) < \infty. \)

4. If \( A \subset \mathbb{R}^n \), \( B \subset \mathbb{R}^n \), and \( \delta = \text{dist}(A, B) > 0 \), then \( U = \{ x : \text{dist}(x, A) < \delta/2 \} \)
and \( V = \{ x : \text{dist}(x, B) < \delta/2 \} \) are disjoint open sets. Choosing, for \( \varepsilon > 0 \) and an open set \( G \)
so that \( \lambda(G) \leq \lambda^*(A \cup B) + \varepsilon \), we conclude that 
\[ \lambda^*(A) + \lambda^*(B) \leq \lambda^*(A) + \lambda^*(B) \leq \lambda(G \cap U) + \lambda(G \cap V) \leq \lambda(G) \leq \lambda^*(A \cup B) + \varepsilon \]
and then let \( \varepsilon \to 0. \)

5. If \( g : [0, 1] \to \mathbb{R}^2 \) satisfies \( |g(s) - g(t)| \leq |s - t| \) for all \( 0 \leq s \leq t \leq 1 \), then the image 
\[ g([0, 1]) = \{ g(t) : t \in [0, 1] \} \]
has 2-dimensional Lebesgue measure zero. In fact, the map \( f : [0, 1] \times \mathbb{R} \to \mathbb{R}^2 \)
\[ f(x, y) = g(x) \]
has \( \text{Lip} f = \text{Lip} g = 1 \) so that 
\[ \lambda_2(g([0, 1])) \leq \lambda_2(f([0, 1] \times \{0\})) \leq (\text{Lip} f)^2 \lambda_2([0, 1] \times \{0\}) \]
\[ = 0. \]
Or more directly, note that for each positive integer \( k \)
\[ g([0, 1]) \subset \bigcup_{j=1}^{k} (g([\frac{j-1}{k}, \frac{j}{k}])) \subset \bigcup_{j=1}^{k} B(g(\frac{j}{k}), \frac{1}{k}) \]
so that \( \lambda_2(g([0, 1])) \leq k \cdot \pi(\frac{1}{k})^2 \to 0 \) as \( k \to \infty. \)
6. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Lebesgue measurable and $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous. The composition $h : \mathbb{R} \to \mathbb{R}$, $h(x) = F(f(x), g(x))$ is measurable. In fact, for any $t \in \mathbb{R}$, $G = F^{-1}((t, \infty))$ is open in $\mathbb{R}^2$. By Problem 3, $G = \bigcup_{i=1}^{\infty} [a_i, b_i] \times [c_i, d_i]$ for some real numbers $a_i, b_i, c_i, d_i$. It follows that the set

$$h^{-1}((t, \infty)) = (f, g)^{-1}(G) = \bigcup_{i=1}^{\infty} (f^{-1}[a_i, b_i] \cap g^{-1}[c_i, d_i]) = \bigcup_{i=1}^{\infty} (f^{-1}[a_i, b_i] \cap g^{-1}[c_i, d_i])$$

is measurable.

7. To find the Lebesgue measure of $g(B)$ where $B$ is the open unit ball in $\mathbb{R}^2$ and $g(x, y) = (x + 2y + 3, x - y - 4)$, we first note, by the translation invariance of Lebesgue measure, that $g(b) = h(B)$ where $h(x, y) = (x + 2y, x - y)$. Next we note that $h([0, 1] \times [0, 1])$ is the parallelogram with vertices $(0, 0), (1, 1), (3, 0), (2, -1)$, and the area of this parallelogram is 3. (The area formula $A = bh$ for a parallelogram is easily checked using the translation and orthogonal invariance of 2 dimensional Lebesgue measure.) It follows, by linearity and translation invariance, that $\lambda(h(Q)) = 3\lambda(Q)$ for any cube $Q \subset \mathbb{R}^2$. Of course, both the boundary of $Q$ and the boundary of $h(Q)$, being contained in 4 lines, has 2 dimensional Lebesgue measure zero. Using Problem 3 and the fact that $h$ is injective, we conclude that $\lambda(h(B)) = 3\lambda(B) = 3\pi$.

8. Suppose $0 < \alpha < \infty$. A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an $\alpha$ similarity if $f(x) - f(0)$ is linear and $|f(x) - f(y)| = \alpha |x - y|$ for all $x, y \in \mathbb{R}^n$.

(a) Show that $\lambda(f(A)) = \alpha^n \lambda(A)$ for any $\alpha$ similarity $f$ and any Lebesgue measurable subset $A$ of $\mathbb{R}^n$. Since Lip $f \leq \alpha$

$$\lambda(f(A)) = \alpha^n \lambda(A).$$

The $\alpha$ similarity definition also implies that $f$ is injective with Lip $f^{-1} \leq 1/\alpha$ so we obtain the opposite inequality

$$\lambda(A) \leq \text{Lip } f^{-1} \lambda(f(A)) \leq \left( \frac{1}{\alpha} \right)^n \lambda(f(A)).$$

(b) Suppose that $E$ is a bounded measurable set and that

$$E = f_1(E) \cup f_2(E) \cup \cdots \cup f_m(E)$$

for some $\alpha$ similarities $f_1, f_2, \cdots, f_m$ of $\mathbb{R}^n$ such that $f_i(E) \cap f_j(E) = \emptyset$ for $1 \leq i < j \leq m$. From (a) and the measurability of $E$ and of each $f_i(E)$, we get the equation

$$\lambda(E) = \lambda(f_1(E)) + \cdots + \lambda(f_m(E)) = ma^n \lambda(E).$$

Here $\lambda(E) < \infty$ because $E$ is bounded. If $\lambda(E) > 0$, then we can cancel it to get the desired equation $\alpha = (1/m)^{1/n}$.
(c) An example with $\lambda(E) = 0$, $m = 2$ and $\alpha = 1/2$ is the “interval” $E = [0, 2) \times \{0\}$ with the 2 similarities

$$f_1(x, y) = \frac{1}{2}(x, y), \ f_2(x, y) = \frac{1}{2}(x + 2, y).$$

An example with $\lambda(E) > 0$, $m = 4$, and $\alpha = (1/4)^{1/2} = 1/2$ is the “square” $E = [0, 2) \times [0, 2)$ with the 4 similarities

$$f_1(x, y) = \frac{1}{2}(x, y), \ f_2(x, y) = \frac{1}{2}(x + 2, y), \ f_3(x, y) = \frac{1}{2}(x, y + 2), \ f_4(x, y) = \frac{1}{2}(x + 2, y + 2).$$

A more interesting example with $\lambda(E) = 0$, $m = 4$, and $\alpha = \frac{1}{3}$ is $E = C \times C$ where $C$ is the standard ternary Cantor set and

$$f_1(x, y) = \frac{1}{3}(x, y), \ f_2(x, y) = \frac{1}{3}(x + 2, y), \ f_3(x, y) = \frac{1}{3}(x, y + 2), \ f_4(x, y) = \frac{1}{3}(x + 2, y + 2).$$

[Here equation (*) just becomes $0 = 0$, but a more useful equation here occurs with $m = 4$, $\alpha = \frac{1}{3}$, $n$ replaced by $\beta = \log 4 / \log 3$, and $n$ dimensional Lebesgue measure $\lambda$ replaced by $\beta$ dimensional “Hausdorff” measure.]

9. Suppose that $A$ and $B$ are Lebesgue measurable subsets of $\mathbb{R}$. To see that $A \times B$ is Lebesgue measurable in $\mathbb{R}^2$ with $\lambda_2(A \times B) = \lambda_1(A)\lambda_1(B)$, we will treat several cases. This is clear in case both $A$ and $B$ are closed intervals. A finite union of closed intervals is a finite disjoint union of closed intervals. It both $A$ and $B$ are such finite disjoint unions, say $A = \bigcup_{i=1}^{m} [a_i, b_i]$ and $B = \bigcup_{j=1}^{n} [c_j, d_j]$, then $A \times B$ is the disjoint union of the rectangles $[a_i, b_i] \times [c_j, d_j]$ and one obtains the measurability. The product formula follows by linearity. If both $A$ and $B$ are open sets, then taking sup’s over enclosed special polygon’s readily gives $\lambda_1(A)\lambda_1(B) \leq \lambda_2(A \times B)$. To verify equality, we observe that $I$ is a special polygon in $A \times B$, then the projection $I_X$ of $I$ onto the X-axis is contained in $A$ and similarly $I_Y \subset B$. So $I \subset I_X \times I_Y \subset A \times B$, and this allows us to get the opposite inequality. For compact sets $A, B$, where the measure is given by the infima over enclosing open sets, we now get immediately the inequality $\lambda_1(A)\lambda_1(B) \geq \lambda_2(A \times B)$. For the opposite inequality, we observe that for any open set $G$ containing the compact set $A \times B$, there are open sets $G_X$ and $G_Y$ with $A \times B \subset G_X \times G_Y \subset G$. Here we can get $G_X$ and $G_Y$ by taking $\delta$ open neighborhoods of $A$ and $B$ where $\sqrt{2}\delta = \text{dist}(A \times B, C^c)$. The general case of measurable $A$ and $B$ now follows by approximation.