Homework 1 due Wednesday, August 31.
Chapter 1: #2(b)(c)(e), #3, #9, #11, #18, #24, #32(e) (In each chapter of the textbook, the problems are numbered, and scattered throughout the text. For example, #2(b)(c)(e) refers to parts (b)(c), and (e) of Problem 2 on page 3.)

Homework 2 due Wednesday, Sept.7.
Chapter 1: #47, #53, #59, #60, Chapter 2: #2, #3, #4(a)(b),
Extra Problem: Prove that the interval $[0,1]$ is not homeomorphic to the solid square $[0,1] \times [0,1]$. (See page 18. Hint: You may use Problem 45. What happens when we remove 1 point? A warning: there actually is a continuous function mapping the interval onto the whole square, but it is not injective.)

Homework 3 due Wednesday, Sept.14.
Chapter 2: #23, #25.
The purpose of the rest of this week’s homework is to show how to get decompositions involving disjoint closed balls (compare Problem #36, Chapter2). This will give us an alternate proof of the rotation invariance of Lebesgue measure.

Problem 3. Show that in $\mathbb{R}^n$, $\left(\frac{2}{\sqrt{n}}\right)^n \leq \lambda(B(0,1)) \leq 2^n$. (The exact value will be found in Chapter 9.)

Problem 4. Show that $\lambda(rG) = r^n \lambda(G)$.

Problem 5. Show that the unit sphere $\partial B(0,1) = \{x : |x| = 1\}$ has $n$ dimensional Lebesgue measure zero. Hint: Use #4 with $G = B(0,1)$.

Problem 6. Show that any open set $G$ equals a countable union of nonoverlapping closed dyadic cubes. Here a closed dyadic cube is a set of the form $2^j Q_k$ where $j \in \mathbb{Z}$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, and $Q_k = [k_1, k_1 + 1] \times \ldots \times [k_n, k_n + 1]$.

Problem 7. Show that any open set $G = Z \cup \bigcup_{j=1}^{N_1} B_j$ where $\lambda(Z) = 0$ and the sets $B_j$ are disjoint closed balls. Hint: Fix a number $\rho$ with $1 - (n) - n/2 < \rho < 1$, and use #1, #2, and #4 to show there are finitely many disjoint closed balls $B_1, \ldots, B_{N_1}$ in $G$ so that $G_1 = G \setminus \bigcup_{j=1}^{N_1} B_j$ has $\lambda(G_1) \leq \rho \lambda(G)$. Then repeat the argument to get disjoint closed balls $B_{N_1+1}, \ldots, B_{N_2}$ in $G_1$ so that $G_2 = G_1 \setminus \bigcup_{j=N_1+1}^{N_2} B_j$ has $\lambda(G_2) \leq \rho \lambda(G_1)$, etc.

Homework 4 due Wednesday, Sept.21.
[Changes: added “linear” assumption to #7,#8 and changed the hints in #3,#6.]

Chapter 2: #28, #34.
One purpose of the rest of this week’s homework is to show the rotation invariance of Lebesgue measure.

Problem 3. Show that $\lambda(Z) = 0$ $\iff$ For every $\epsilon > 0$ there are open balls $B_i$ with $Z \subset \bigcup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} \lambda(B_i) < \epsilon$. [Hint: For the implication one can first choose an open
$U$ containing $Z$ with $\lambda(U) < \epsilon/(2\sqrt{n})^n$, then use Problem 6 from last week and note that any closed cube $Q$ is contained in an open ball $B$ with $\lambda(B) < (2\sqrt{n})^n \lambda(Q)$.]

**Problem 4.** A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz if the number

$$\text{Lip } f \equiv \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

is finite. Show that for any open ball $B$, $\lambda(f(B)) \leq (\text{Lip } f)^n \lambda(B)$. [Hint: Recall Problem 4 from last week.]

**Problem 5.** Show that $\lambda(f(Z)) = 0$ for any Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^n$ and any set $Z \subset \mathbb{R}^n$ with $\lambda(Z) = 0$.

**Problem 6.** Show that $\lambda(f(G)) \leq (\text{Lip } f)^n \lambda(G)$ for any open subset $G$ of $\mathbb{R}^n$. [Hint: Use Problem 7 from last week.]

**Problem 7.** A linear map $g : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if $g(x) \cdot g(y) = x \cdot y$ for all $x, y \in \mathbb{R}^n$. Show that such a map is injective and surjective with $\text{Lip } g = 1$ and $\text{Lip } g^{-1} = 1$. [Hint: $d(x, y)^2 = |x - y|^2 = x \cdot x - 2x \cdot y + y \cdot y$.]

**Problem 8.** (Rotation Invariance) Show that $\lambda(g(A)) = \lambda(A)$ for any orthogonal linear map $g : \mathbb{R}^n \to \mathbb{R}^n$ and any Lebesgue measurable $A \subset \mathbb{R}^n$.

**Homework 5** due Monday, Oct.3 **NOTE THIS NEW DATE BECAUSE OF RITA.**

- Chapter 2: #32, #35, #42, #45, #46, #47 [Hint: use #46].
- Chapter 4: #2.

**Problem 8.** (outer measure) Suppose $\mu(E) \in [0, \infty]$ for all $E \subset \mathbb{R}^n$, $\mu(\emptyset) = 0$, and $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$ whenever $E_i \subset \mathbb{R}^n$. One says that a subset $A$ of $\mathbb{R}^n$ is $\mu$ measurable if

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A) \quad \text{for all } E \subset \mathbb{R}^n.$$  

Show that $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ whenever $A_i$ are disjoint $\mu$ measurable subsets of $\mathbb{R}^n$. [Hint: Prove the superadditivity first for 2 disjoint $\mu$ measurable sets, then inductively for finitely many, then take a limit.]

**Homework 6** due Wed., Oct.12 (after break)

- Chapter 4: #7,
- Chapter 5: #2, #8, #10, #11, #13, #14.

**Homework 7** due Wed., Oct.19

- Chapter 5: #16, #18, #20, #23.
- Chapter 6: #1, #5, #8.

**Problem 8.** Prove that for every measurable simple function $s$ on $\mathbb{R}^n$ there exists a sequence of continuous functions $f_i : \mathbb{R}^n \to \mathbb{R}$ so that

$$\lim_{i \to \infty} f_i(x) = s(x) \quad \text{for almost every } x \in \mathbb{R}^n.$$  

[Hint: See the Theorem in Chapter 1, Section F.]
Homework 8 due Mon., Oct.31

Chapter 6: #12, #19, #21, #22, #24, #36,

Problem 7. Suppose that $f \in L^1(E)$ where $E$ is a measurable set of finite positive measure. Show that there is a point $x \in E$ with $|f(x)| \leq (\lambda(E)^{-1} \int_E |f| \, d\lambda)$.

Problem 8. (Corrected!) Suppose $f, f_1, f_2, f_3, \ldots$ are all $L^1$ functions on $\mathbb{R}^n$ and that $\int |f_k - f| \, d\lambda \to 0$ as $k \to \infty$. Prove that for every $\varepsilon > 0$ there exists an integer $k_\varepsilon$ so that

$$\lambda\{x : |f_k(x) - f(x)| > \varepsilon\} < \varepsilon$$

whenever $k \geq k_\varepsilon$.

Homework 9 due Mon., Nov.7

Chapter 6: #28, #30, #35, #40, #44, #45,

Problem 7. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. For $E \subset X$, we define

$$\nu(E) = \inf \{\mu(A) : E \subset A \in \mathcal{M}\}.$$

Prove that $\nu$ is an outer measure on $X$ (i.e. countably subadditive) and that one has the equality

$$\nu(E) = \nu(E \cap A) + \nu(E \setminus A) \quad \text{for all } A \in \mathcal{M}.$$

(Thus every measurable set for $\mu$ is also measurable for $\nu$.) [Hint you may use some results from the book for Lebesgue measure if you just check that the proofs carry over to the general measure $\mu$.]

Homework 10 due Mon., Nov.14

Chapter 6: #32, #36, #39, #42, Chapter 7: #22

Problem 6. (Corrected version: I changed a 2 to $\frac{1}{2}$ in the definition of $f_j$) For each positive integer $j$, let

$$f_j = \frac{1}{2^j} \cdot \chi_{[-1/j, 1/j]}$$

and $\nu_j(E) = \int_E f_j \, d\lambda$ for Lebesgue measurable $E \subset \mathbb{R}$. 

1. Show that the measures $\nu_j$ approach $\delta_0$ as $j \to \infty$ in the sense that

$$\int g \, d\nu_j \to \int g \, d\delta_0 \quad \text{for every continuous } g : \mathbb{R} \to \mathbb{R}.$$

2. Is it true that $\nu_j(E) \to \delta_0(E)$ for all Borel subsets $E$ of $\mathbb{R}$? Prove this or find an example of a Borel $E$ where this isn’t true.
Homework 10 due Mon., Nov. 21 (with 2 corrections. \( B \to A \) in Problem 2(2) and \(|b_i - a_i| \leq \delta \) added in the definition in Problem 3.)

Chapter 7: #20.

Problem 2. For any 2 Lebesgue measurable subsets \( A, B \subset \mathbb{R}^n \), define

\[
dist_\lambda(A, B) = \lambda(A \setminus B) + \lambda(B \setminus A).
\]

(1) Show that the \( \dist_\lambda(A, C) \leq \dist_\lambda(A, B) + \dist_\lambda(B, C) \).

(2) Prove that \( \lim_{c \to 0} \dist_\lambda(A + c, A) = 0 \) where \( A + c = \{a + c : a \in A\} \).

Problem 3. For \( 0 \leq r < \infty \) and \( A \subset \mathbb{R} \), we define the Hausdorff outer measure

\[
\mathcal{H}^r(A) = \lim_{\delta \to 0} \inf \{ \sum_{i=1}^\infty (b_i - a_i)^r : A \subset \bigcup_{i=1}^\infty [a_i, b_i], \ |b_i - a_i| \leq \delta \}.
\]

(1) Prove that if \( \mathcal{H}^r(A) < \infty \), then \( \mathcal{H}^s(A) = 0 \) for all \( s > r \).

(2) Prove that if \( C \) is the standard tertiary Cantor set, then \( \mathcal{H}^t(C) < \infty \) where \( t = \log 2 / \log 3 \).

Homework 11 due Wed., Nov. 30

Chapter 8: #1, #2, #7, #14, #16, #19

Chapter 10: #1, #21 [Hint: Use Hölder's inequality.]