Assignmen t 6, due 10/29

Let $C_0^\infty(R) = \{\text{infinitely differentiable } f : R \to R \text{ of compact support} \}$. The weak derivative of a locally integrable function $f : R \to R$ is the linear function $D_f : C_0^\infty(R) \to R$ defined by

$$D_f(g) = -\int fg' \, dx \text{ for all } g \in C_0^\infty(R).$$

**Ex. 1** Find another formula for $D_f(g)$ in case $f$ is continuously differentiable.

**Ex. 2** Find another formula for $D_A(g)$ where $A(x) = |x|$.

**Ex. 3** Find another formula for $D_H(g)$ where $H(x) = 0$ for $x \leq 0$ and $H(x) = 1$ for $x > 0$.

**Ex. 4** Let $\gamma(x, y) = \lambda(|x - y|)H(y - x)$ where $\lambda(r)$ is a cutoff function as on P.297 of Royden’s paper. Let

$$(T\phi)(y) = \int \phi(x)\gamma(x, y) \, dx \text{ for } \phi \in C_0^\infty(R).$$

Arguing now as in proof of Lemma 1 in Royden’s paper, show that $T\phi$ is differentiable and (using Ex.3) that

$$(T\phi)'(y) = \phi(y) + \int_{-\infty}^y \phi(x)\lambda'(y - x) \, dx = \phi(y) - \int \phi(x)\gamma_x(x, y) \, dx.$$  

Hint: You may use the fact that $G(y) = \int_{-\infty}^y F(x, y) \, dx$ is smooth whenever $F$ is smooth and that $G'(y) = F(y, y) + \int_{-\infty}^y F_y(x, y) \, dx$. This fact follows by applying the Fundamental Theorem of Calculus and the chain rule to differentiate $G(y) = H(y, y)$ where $H(y, z) = \int_{-\infty}^y F(x, z) \, dx$.

**Ex. 5** Using Ex.4 and arguing as in the proof of Lemma 1, show:

If $D_f \in C^\infty(R)$ (that is, for some $h \in C^\infty(R)$, $D_f(g) = \int g \cdot h \, dx$ for all $g \in C_0^\infty(R)$, then $f \in C^\infty(R)$.

Hint: Take $g = T\phi$, change variables to obtain the common factor $\phi(x) \, dx$ in all integrals, and use Fubini’s Theorem to find the equation

$$-\int_{-\infty}^\infty \lambda(y - x)h(y) \, dy = f(x) + \int_{-\infty}^\infty f(y)\lambda'(y - x) \, dy,$$

which shows the smoothness of $f$. 

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