1. (a) The Cauchy-Riemann equations imply that
\[ u_x = v_y = -2y(3x + 1), \quad u_y = -v_x = -3x^2 - 2x + 3y^2. \]
Integrating we find that \( u = -3x^2y - 2xy + y^3. \).
(b) The general solution is \( u = -3x^2y - 2xy + y^3 + c \) for some constant \( c \), because the difference of any two solutions has by the Cauchy-Riemann equations, gradient zero. So the difference must be a constant.

2. Suppose that \( g \) is twice continuously differentiable and real-valued on \( \mathbb{R}^2 \). You are to prove that
\[ \frac{\partial^2 g}{\partial x \partial y}(0,0) = \frac{\partial^2 g}{\partial y \partial x}(0,0), \quad (*) \]
using the following steps:
(a)
\[ \int_0^b \int_0^a \frac{\partial^2 g}{\partial x \partial y}(x,y) \, dx \, dy = \int_0^b \left[ \frac{\partial g}{\partial y}(a,y) - \frac{\partial g}{\partial y}(0,y) \right] \, dy = g(a,b) - g(a,0) - g(0,b) + g(0,0). \]
(b) Using Fubini's Theorem, we also find that
\[ \int_0^b \int_0^a \frac{\partial^2 g}{\partial y \partial x}(x,y) \, dx \, dy = \int_0^a \int_0^b \frac{\partial^2 g}{\partial y \partial x}(x,y) \, dy \, dx = \int_0^a \left[ \frac{\partial g}{\partial x}(x,b) - \frac{\partial g}{\partial x}(x,0) \right] \, dx = g(a,b) - g(0,b) - g(a,0) + g(0,0). \]
(c) (a) and (b) are clearly the same.
(d) Using the Fundamental Theorem of Calculus, we conclude
\[ \frac{\partial^2 g}{\partial x \partial y}(0,0) = \lim_{a \to 0} \frac{1}{a} \int_0^a \frac{\partial^2 g}{\partial x \partial y}(x,0) \, dx \]
\[ = \lim_{a \to 0} \frac{1}{a} \int_0^b \lim_{a \to 0} \frac{1}{a} \int_0^a \frac{\partial^2 g}{\partial x \partial y}(x,y) \, dx \, dy \]
\[ = \lim_{b \to 0} \frac{1}{b} \int_0^b \lim_{a \to 0} \frac{1}{a} \int_0^a \frac{\partial^2 g}{\partial y \partial x}(x,y) \, dx \, dy \]
\[ = \lim_{a \to 0} \frac{1}{a} \int_0^a \frac{\partial^2 g}{\partial y \partial x}(x,0) \, dx \]
\[ = \frac{\partial^2 g}{\partial y \partial x}(0,0). \]
3. Suppose that $D = \{ z \in \mathbb{C} : |z| < 1 \}$, $f : D \to D$ is holomorphic, and $z_0 \in D$. Let $w_0 = f(z_0)$. Let
\[ F(z) = \frac{z - z_0}{1 - \bar{z}_0 z}, \quad G(w) = \frac{w - w_0}{1 - \bar{w}_0 w}, \quad g(\zeta) = (G \circ f \circ F^{-1})(\zeta). \]
Then, $g : D \to D$ is holomorphic with $g(0) = (G \circ f)(z_0) = G(w_0) = 0$. Applying the Schwarz Lemma to $g$, we conclude that $|g(\zeta)| \leq |\zeta|$ for all $\zeta \in D$. So, with $\zeta = F(z)$,
\[ \left| \frac{f(z) - w_0}{1 - \bar{w}_0 f(z)} \right| = |G \circ f(z)| = |g(\zeta)| \leq |\zeta| = \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|. \]
(b) From (a) we have that
\[ \left| \frac{f(z) - w_0}{1 - \bar{w}_0 f(z)} \right| \leq \frac{1}{|1 - \bar{z}_0 z|}. \]
Taking the limit as $z \to z_0$, and noting that $w_0 = f(z_0)$ and that both $1 - |f(z_0)|^2$ and $1 - |z_0|^2$ are positive, we conclude that
\[ \left| \frac{f'(z_0)}{1 - |f(z_0)|^2} \right| \leq \frac{1}{1 - |z_0|^2}, \]
and then replace $z_0$ by $z$.

4. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that, for almost all $t \in \mathbb{R}$, $f'(t)$ exists and $|f'(t)| \leq 1$. Is it true that, $f(b) - f(a) = \int_a^b f'(t) \, dt$ for $-\infty < a < b < \infty$?

No. If $f(t)$ is the Cantor function for $0 \leq t \leq 1$, $f|(-\infty, 0] \equiv 0$, and $f|[0, +\infty) \equiv 0$, the $f$ is continuous with $f'(t) = 0$ for a.e. $t$, but $f(1) \neq f(0)$.

(b) Suppose $g : \mathbb{R} \to \mathbb{R}$ is differentiable at every point $t \in \mathbb{R}$. Is $g$ necessarily of bounded variation on every closed interval $[a, b] \subset \mathbb{R}$?

No, we can define $g(0) = 0$ and $g(t) = t^2 \cos(2\pi t^2)$ for $t \neq 0$. Here $g'(0) = 0$ because $|g(t)| \leq t^2$ and, for $t \neq 0$, $g'(t)$ exists by the product and chain rules. Taking $t_n = n^{-1/2}$, we find that the variation of $g$ is infinite on any interval containing 0 because $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty$.

5. Suppose that $f$ is a holomorphic function on the punctured plane $\mathbb{C} \setminus \{0\}$.

(a) For each positive numbers $\varepsilon < R < \infty$, find a formula for $f(z)$ on the annulus $\{ z \in \mathbb{C} : \varepsilon < |z| < R \}$ in terms of the values of $f$ on the inner boundary circle $\{ z : |z| = \varepsilon \}$ and on the outer boundary circle $\{ z : |z| = R \}$. To get a suitable contour of integration, we may remove any thin radial strip from the annular region $\{ z \in \mathbb{C} : \varepsilon < |z| < R \}$. Applying the Cauchy integral formula on the boundary of this region and then letting the width of the thin strip approach 0, we conclude that
\[ f(z) = \frac{1}{2\pi i} \left[ \int_{\partial B_R} \frac{f(\zeta)}{z - \zeta} d\zeta - \int_{\partial B_\varepsilon} \frac{f(\zeta)}{z - \zeta} d\zeta \right]. \]
(b) If \( f \) is meromorphic and
\[
\int_{\{z: 0 < |z| < 1\}} |f(z)| \, dx \, dy < \infty , 
\]
then, at 0, \( f \) either has a removable singularity or is meromorphic with a pole of order 1. Since \( f \) is meromorphic, one has, on a punctured neighborhood of the origin, \( z^k f(z) = g(z) \), for some nonnegative integer \( k \) and nonvanishing holomorphic function \( g \). If \( k \geq 2 \), then, for \( |z| < \epsilon \) with \( \epsilon \) sufficiently small,
\[
\frac{1}{2} \frac{|g(0)|}{|z|^k} < |f(z)| < 2 \frac{|g(0)|}{|z|^k} .
\]
Also
\[
\int_{B_\epsilon} |z|^{-k} \, dx \, dy = \int_0^{2\pi} \int_0^{\epsilon} r^{1-k} \, dr \, d\theta < \infty
\]
if and only if \( k < 2 \). For \( k = 0 \) the singularity is removable. For \( k = 1 \), it is a pole of order 1.

(c) Does the integrability assumption (***) alone imply that \( f \) is automatically meromorphic at 0. Yes, for any \( r > 0 \) we can choose, by Fubini’s Theorem, a number \( \epsilon(r) \in [\frac{r}{2}, r] \) so that
\[
\int_{\partial B_{\epsilon(r)}} |f| \, \leq \frac{2}{r} \int_{B_r} |f| \, dx \, dy .
\]
It follows that for fixed \( z \) the line integral
\[
|\int_{\partial B_{\epsilon(r)}} \frac{\zeta f(\zeta)}{z - \zeta} \, d\zeta| \leq \frac{2}{\text{dist}(z, \partial B_{\epsilon(r)})} \int_{B_\epsilon} |f| \, dx \, dy \to 0 \text{ as } r \to 0 .
\]

Applying the formula from (a) with \( f(z) \) replaced by \( zf(z) \), we conclude that \( zf(z) \) has a removable singularity at 0, so that \( f \) is meromorphic with a pole of order \( \leq 1 \) at 0.

6. Suppose that \( E_1, E_2, E_3, \ldots \) is a sequence of Lebesgue measurable subsets of the unit ball \( B \) in \( \mathbb{R}^n \), and that each \( E_k \) has positive Lebesgue measure \( \mu(E_k) > \varepsilon \) for a fixed \( \varepsilon > 0 \). For each \( x \in B \), let \( n(x) \) denote the number of integers \( k \) so that \( x \in E_k \).

(a) Show that \( n(x) \geq 2 \) for some \( x \in B \). Otherwise, the \( E_k \) are disjoint and
\[
\infty > \mu(B) \geq \mu(\cup_{k=1}^\infty E_k) = \sum_{k=1}^\infty \mu(E_k) = \infty .
\]

(b) Show that \( \sup_{x \in B} n(x) = \infty \).
\[
\infty = \sum_{k=1}^\infty \int \chi_{E_k} \leq \int \sum_{k=1}^\infty \chi_{E_k} \leq [\sup_{x \in B} n(x)] \mu(B) .
\]
(c) Show that $n(x) = \infty$ for some $x \in B$.

The sets $F_j = \bigcup_{k=j}^{\infty} E_k$, form a decreasing sequence of measurable subsets of the finite measure set $B$. So $F = \bigcap_{j=1}^{\infty} F_j$ is measurable with

$$
\mu(F) = \lim_{j \to \infty} \mu(F_j) \geq \lim_{j \to \infty} \mu(E_j) = \varepsilon > 0 .
$$

So $F$ contains a point $x$. Since $x \in F_1$, $x \in E_{n(1)}$ for some positive integer $n(1)$. Since $x \in F_{n(1)+1}$, $x \in E_{n(2)}$ for some integer $n(2) > n(1)$. Continuing, we inductively find a sequence of integers $n(1) < n(2) < n(3) < \ldots$ so that $x \in \bigcap_{i=1}^{\infty} E_{n(i)}$. 
