HW#15 §3.9 #2, 6, 12, 24, 28, 38, 52, 60

In Problems 1 through 4, first find the derivative \( \frac{dy}{dx} \) by implicit differentiation. Then solve the original equation for \( y \) explicitly in terms of \( x \) and differentiate to find \( \frac{dy}{dx} \). Finally verify that your two results are the same by substitution of the explicit expression for \( y(x) \) in the implicit form of the derivative.

2) \( xy=1 \)

Differentiating implicitly \( y + x \frac{dy}{dx} = 0 \) or \( \frac{dy}{dx} = -\frac{y}{x} \). Solving for \( y \) gives \( y = \frac{1}{x} \) and therefore \( \frac{dy}{dx} = -\frac{1}{x^2} \). Plugging \( y = \frac{1}{x} \) into the implicit formula gives \( \frac{dy}{dx} = -\frac{y}{x} = -\frac{1}{x^2} \).

In Problems 5 through 14, find \( \frac{dy}{dx} \) by implicit differentiation.

6) \( x^4 + x^2 y^2 + y^4 = 48 \)

Implicit differentiation gives \( 4x^3 + 2x y^2 + 2x^2 y \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = 0 \) or \( \frac{dy}{dx} = -\frac{4x^3 + 2x y^2}{2x^2 y + 4y^3} \).

12) \( \cos(x+y) = \sin x \sin y \)

Implicit differentiation gives \( -\sin(x+y) \left( 1 + \frac{dy}{dx} \right) = \cos x \sin y + \sin x \cos y \frac{dy}{dx} \) or \( \frac{dy}{dx} = -\frac{\sin(x+y) + \cos x \sin y}{\sin(x+y) + \sin x \cos y} \).

In Problems 15 through 28, use implicit differentiation to find an equation of the tangent line to the given curve at the given point.

24) \( xy = 6 e^{2x-3y} \), \( (3,2) \)

Implicit differentiation gives \( y + x \frac{dy}{dx} = 6 e^{2x-3y} \left( 2 - 3 \frac{dy}{dx} \right) \) or \( \frac{dy}{dx} = \frac{12 e^{2x-3y} - y}{18e^{2x-3y} + x} \).

Therefore \( \frac{dy}{dx}_{(3,2)} = \frac{12 - 2}{18 + 3} = \frac{10}{21} \). Since the tangent line passes through \( (3,2) \) its equation is \( (y-2) = \frac{10}{21}(x-3) \) or \( y = \frac{10}{21}x + \frac{4}{7} \).
Implicit differentiation gives \[ 2y \frac{dy}{dx} = 3x^2 + 14x \] or \[ \frac{dy}{dx} = \frac{3x^2 + 14x}{2y}. \] Hence at \( x = -3 \) and \( y = 6 \) we get \[ \frac{dy}{dx} = -\frac{5}{4}. \] Thus the equation is \( (y-6) = -\frac{5}{4}(x+3) \) or \( y = -\frac{5}{4}x + \frac{9}{4}. \)

38) Suppose that water is being emptied from a spherical tank of radius 10 ft. If the depth of the water in the tank is 5 ft and is decreasing at the rate of 3 ft/sec, at what rate is the radius of the top surface of the water decreasing?

Let \( y \) be the height of the water in feet and \( r \) the radius of the top surface in feet. We are told \( \frac{dy}{dt} = -3 \) and we are asked for \( \frac{dr}{dt} \). Since there is a right triangle with sides \( 10 - y \) and \( r \) and hypotenuse 10, we have \( (10- y)^2 + r^2 = 100 \). Differentiating gives \( -2(10-y) \frac{dy}{dt} + 2r \frac{dr}{dt} = 0 \) or \( \frac{dr}{dt} = \frac{10-y}{r} \frac{dy}{dt} \). When \( y = 5 \), \( 5^2 + r^2 = 100 \) or \( r = 5 \sqrt{3} \), thus \( \frac{dr}{dt}_{y=5} = \frac{10 - 5}{5 \sqrt{3}} \cdot (-3) = -\sqrt{3} \). Thus the radius is decreasing at \( \sqrt{3} \) ft/sec.

52) The base of a rectangle is increasing at 4 cm/s while its height is decreasing at 3 cm/s. At what rate is its area changing when its base is 20 cm and its height is 12 cm?

Let \( b \) be the length of the base in cm and \( h \) the height in cm. We are told that \( \frac{db}{dt} = 4 \) and \( \frac{dh}{dt} = -3 \). Since the area is given by \( A = bh \), when \( b = 20 \) and \( h = 12 \) we have

\[
\frac{dA}{dt} = h \frac{db}{dt} + b \frac{dh}{dt} = (12)(4) + (20)(-3) = 48 - 60 = -12.
\]

Therefore the area is decreasing by 12 cm²/sec.
A ship with a long anchor chain is anchored in 11 fathoms of water. The anchor chain is being wound in at a rate of 10 fathoms/min, causing the ship to move toward the spot directly above the anchor resting on the seabed. The hawsehole—the point of contact between the ship and chain—is located 1 fathom above the water line. At what speed is the ship moving when there are exactly 13 fathoms of chain still out.

Let \( L \) be the length of chain out (in fathoms) and let \( x \) be the horizontal distance from the ship to spot directly above the anchor (also in fathoms). Let \( t \) be time measured in minutes. Then we are told that \( \frac{dL}{dt} = -10 \) and we are asked to find \( \frac{dx}{dt} \).

Because the anchor chain forms the hypotenuse of a right triangle with one (horizontal) side \( x \) and the other (vertical) side \( 11 + 1 = 12 \), we see that

\[
L^2 = x^2 + 12^2 = x^2 + 144.
\]

Differentiating with respect to \( t \) gives

\[
2L \frac{dL}{dt} = 2x \frac{dx}{dt}
\]

or

\[
\frac{dx}{dt} = \frac{L}{x} \frac{dL}{dt}.
\]

At the time when \( L = 13 \), \( 13^2 = x^2 + 144 \) hence \( x^2 = 25 \) or \( x = 5 \). Thus at this time \( \frac{dx}{dt} = \frac{13}{5}(-10) = -26 \). Thus the ship is moving toward the spot directly above the anchor at 26 fathoms/min.