1. (15 points) Find the solution of the initial value problem

\[ t^2 y' + 2ty = e^{-t}, \quad y(1) = 1. \]

Determine the interval of existence of this solution.

**Answer.** Divide by \( t^2 \) to bring the equation into the form:

\[ y' + \frac{2}{t}y = \frac{e^{-t}}{t^2}. \]

It is a first-order linear ODE. First, we compute the integrating factor:

\[ u(t) = e^{\int \frac{2}{t} dt} = e^{2\ln|t|} = t^2. \]

The general solution is given by

\[ y(t) = \int \frac{u(t)}{u(t)} e^{-t} dt = \int \frac{e^{-t} dt}{t^2} = -e^{-t} + C/t^2. \]

From \( y(1) = 1 \), we find that \( C = 1 + e^{-1} \), hence

\[ y(t) = -e^{-t} + \frac{1 + e^{-1}}{t^2}. \]

The solution is defined if \( t \neq 0 \), hence the interval of existence is \((0, \infty)\).

2. (15 points) Find the solution of the initial value problem

\[ y' = y^2 \cos x, \quad y(0) = 1. \]

What is the interval of existence on which the solution is defined?

**Answer.** Separate variables: \( \frac{dy}{y^2} = \cos(x)dx \). Integrate and obtain:

\[ -\frac{1}{y} = \sin(x) + C. \]

From \( y(0) = 1 \), find \( C \): \(-1 = 0 + C \Rightarrow C = -1 \). The explicit solution is

\[ y(x) = \frac{1}{1 - \sin x}. \]

The solution is well-defined if \( \sin x \neq 1 \), i.e. \( x \neq \frac{\pi}{2} + 2k\pi \). The interval of existence is \((-3\pi/2, \pi/2)\).

3. (15 points) A tank contains 200 gal of salt-water solution with 0.25 lb of salt per gallon. Water containing 1 lb of salt per gallon is poured into the tank at a rate of 2 gal/min and the mixture is drained from the tank at the same rate.

(a) Find the amount of salt \( Q(t) \) in the tank at any time.
(b) Find the moment of time \( T \) when the *concentration* of salt in the tank is 0.5 lb/gal. The final answer may involve a logarithm function.
(c) Find the limiting *concentration* of salt in the tank.
Answer. (a) The balance law says that
\[
\frac{dQ}{dt} = \text{rate in} - \text{rate out} = 1 \text{ lb/gal} \cdot 2 \text{ gal/min} - \frac{Q(t)}{200} \text{ lb/gal} \cdot 2 \text{ gal/min}.
\]
We need to solve the following IVP:
\[
\frac{dQ}{dt} + \frac{1}{100}Q = 2, \quad Q(0) = 200 \cdot 0.25 = 50.
\]
This is a first-order linear equation; first, we compute the integrating factor:
\[
u(t) = e^{\int \frac{1}{100} dt} = e^{t/100}.
\]
The general solution is:
\[
Q(t) = \int 2 \cdot e^{t/100} \, dt - \frac{200e^{t/100} + C}{e^{t/100}} = 200 + Ce^{-t/100}.
\]
From \(Q(0) = 50\) we find that \(200 + C = 50\), hence \(C = -150\). The amount of salt at time \(t\) is
\[
Q(t) = 200 - 150e^{-t/100}.
\]
(b) We need to find \(T\) so that \(\frac{Q(T)}{200} = 0.5\), i.e. \(200 - 150e^{-T/100} = 100\). Solving for \(T\), we get \(T = 100 \ln(3/2)\).
(c) The limiting concentration is
\[
\lim_{t \to \infty} \frac{Q(t)}{200} = \lim_{t \to \infty} \frac{200 - 150e^{-t/100}}{200} = 1 \text{ lb/gal}.
\]
4. (15 points) Consider the first order autonomous differential equation
\[
y' = (y + 1)^2(y - 1)(y - 2).
\]
(a) Sketch the graph of \(f(y) = (y + 1)^2(y - 1)(y - 2)\). Clearly label the equilibrium points.
(b) Draw a phase line and classify each equilibrium point as either unstable or asymptotically stable.
(c) Sketch the equilibrium solutions in the \(ty\)-plane. Sketch at least one solution trajectory in each of the regions of the \(ty\)-plane divided by the equilibrium solutions.

Answer. The equilibrium solutions are \(y = -1\) (unstable), \(y = 1\) (asymptotically stable) and \(y = 2\) (unstable).
5. (10 points) Suppose that \( y(t) \) is a solution of the initial value problem
\[
y' = (y^2 - 4)e^{ty}, \quad y(1) = 1.
\]
Show that \(-2 < y(t) < 2\) for all \( t \) for which \( y \) is defined.

Answer. Notice that \( y(t) = 2 \) and \( y(t) = -2 \) are constant solutions of the ODE. Moreover, the function \( f(t, y) = (y^2 - 4)e^{ty} \) is continuous and the partial derivative \( \frac{\partial f}{\partial y} = 2ye^{ty} + (y^2 - 4)te^{ty} \) is also continuous. We can apply the uniqueness theorem to conclude that the solution curve starting at \( y(1) = 1 \) cannot cross the solution lines \( y = 2, \ y = -2 \), hence
\[
-2 < y(t) < 2.
\]

6. (15 points) Mark opens a savings account with an initial deposit of $1000. The annual interest rate is 4% compounded continuously. He pledges to deposit each year 5% of his salary. Given that his current salary is $35,000 and that he expects his salary to increase continuously at a rate of 2% per year, determine the initial value problem that models the account balance \( P(t) \) at any time \( t \). (Do not solve the equation.)

Answer. Mark’s salary \( S(t) \) at time \( t \) is given by:
\[
\frac{dS}{dt} = 0.02S(t),
\]

hence \( S(t) = S_0e^{0.02t} = 35,000e^{0.02t} \). The IVP problem modeling the account balance \( P(t) \) is
\[
\frac{dP}{dt} = 0.04P + 0.05 \times 35,000e^{0.02t}, \quad P(0) = 1000.
\]

7. (15 points) Consider the second order linear differential equation
\[
y'' - 2y' - 3y = 0.
\]

(a) Find the general solution.

(b) Suppose \( y(0) = 1 \) and \( y'(0) = \alpha \). Find \( \alpha \) so that the solution remains finite as \( t \to \infty \).

Answer. (a) The characteristic equation is \( \lambda^2 - 2\lambda - 3 = 0 \) with roots \( \lambda_1 = 3 \) and \( \lambda_2 = -1 \). Therefore, the general solution of this second order linear ODE is
\[
y(t) = C_1e^{3t} + C_2e^{-t}.
\]
(b) We find the solution that corresponds to $y(0) = 0$, $y'(0) = \alpha$. This implies that $C_1 + C_2 = 1$ and $3C_1 - C_2 = \alpha$. Solving for $C_1$ and $C_2$ one gets

$$C_1 = \frac{\alpha + 1}{4}, \quad C_2 = \frac{3 - \alpha}{4}.$$

Thus

$$y(t) = \frac{\alpha + 1}{4} e^{3t} + \frac{3 - \alpha}{4} e^{-t}.$$

In order for $y(t)$ to be finite as $t \to \infty$, we need

$$\frac{\alpha + 1}{4} = 0 \Rightarrow \alpha = -1.$$