Homework #13 Due April 23.

1. Does there exist a function \( f(z) \) that is holomorphic near the origin and that satisfies
   \[ f(1/n) = f(-1/n) = \frac{1}{n^3}, \quad \text{for } n = 1, 2, \ldots ? \]
   Why or why not?

2. Suppose that
   \[ f(z) = a_0 + a_1 z + a_2 z^2 + \ldots \quad \text{and} \]
   \[ g(z) = b_{-2} z^{-2} + b_{-1} z^{-1} + b_0 + b_1 z + b_2 z^2 + \ldots, \]
   where the two series \( \sum_{n=0}^{\infty} a_n z^n \) and \( \sum_{n=0}^{\infty} b_n z^n \) converge for \( |z| < 2 \). Find \( \int_{\Gamma} f(z) g(z) \, dz \) in terms of \( a_n \) and \( b_n \), where \( \Gamma \) is the positively oriented unit circle.

3. (a) Prove that if \( f \) is a holomorphic map from the unit disk \( D = \{ z \mid |z| < 1 \} \) to itself with \( f(0) = 0 \), then \( |f(z)| \leq |z| \) for all \( z \in D \).
   (b) Which of these \( f \) admit a point \( a \neq 0 \) with \( |f(a)| = |a| \)?
   (c) Let \( g \) be a holomorphic map of the disk \( D \) to itself which is not the identity map of \( D \). Show that \( g \) can have at most one fixed point (i.e., a point \( a \in D \) with \( g(a) = a \)).

4. Suppose that \( f \) is a holomorphic function on \( \{ z \mid |z| < 3 \} \), and that \( f(0) = 0 \). Let \( M_R = \sup_{|z| \leq R} |f(z)| \), and \( N_R = \sup_{|z| \leq R} |f'(z)| \).
   (a) Estimate \( M_R \) (from above) in terms of \( N_R \).
   (b) Estimate \( N_R \) (from above) in terms of \( M_R \).

5. Let \( f \) be an analytic function such that \( f(z) = 1 + z + z^2 + \cdots \) for \( |z| < 1 \). Define a sequence of real numbers \( a_0, a_1, a_2, \ldots \) by
   \[ f(z) = \sum_{n=0}^{\infty} a_n (z + 2)^n. \]
   What is the radius of convergence of this new series \( \sum_{n=0}^{\infty} a_n (z + 2)^n \)?

6. Prove that there is no one-to-one conformal map of the punctured disk \( G = \{ z \in \mathbb{C} \mid 0 < z < 1 \} \) onto the annulus \( A = \{ z \in \mathbb{C} \mid 1 < |z| < 2 \} \).