Math 211 Sec. 4

Models of Motion

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Models of Motion

History of models of planetary motion
• Babylonians - 3000 years ago
  • Initiated the systematic study of astronomy.
  • Collection of astronomical data.

Greeks
• Descriptive model - Ptolemy (~ 100)
  • Geocentric model
  • Epicycles
• Enabled predictions
• No causal explanation
• This model was refined over the following 1400 years.
**Nicholas Copernicus (1543)**
- Shifted the center of the universe to the sun.
- Fewer epicycles required.
- Still descriptive and not causal.
- The shift to a sun centered universe was a major change in human understanding of their place in the universe.

**Johann Kepler (1609)**
- Based on experimental work of Tycho Brahe (1400).
- Three laws of planetary motion.
  1. Each planet moves in an ellipse with the sun at one focus.
  2. The line between the sun and a planet sweeps out equal areas in equal times.
  3. The ratio of the semi-major axis to the cube of the period is the same for each planet.
- This model was still descriptive and not causal.

**Isaac Newton**
- Three major contributions.
  - Laws of mechanics.
    - Second law — $F = ma$.
  - Universal law of gravity.
  - Fundamental theorem of calculus.
    - $f' = g \Leftrightarrow \int g(x) \, dx = f(x) + C$.
  - Invention of calculus.
  - *Principia Mathematica* 1687
Isaac Newton (cont.)

- Laws of mechanics and gravitation were based on his own experiments and his understanding of the experiments of others.
- Derived Kepler's three laws of planetary motion.
- This was a causal explanation.
  - For any mechanical motion.
  - Still used today.

Isaac Newton (cont.)

- Problems with Newton's theory.
  - The force of gravity was action at a distance.
  - Physical anomalies.
    - The Michelson-Morley experiment (1881-87).

Albert Einstein

- Special theory of relativity – 1905.
- General theory of relativity – 1916.
  - Gravity is due to curvature of space-time.
  - Curvature of space-time is caused by mass.
  - Gravity is no longer action at a distance.
- All known anomalies explained.
Unified Theories

- Four fundamental forces.
  - Gravity, electromagnetism, strong nuclear, and weak nuclear.
- Last three can be unified by quantum mechanics. — Quantum chromodynamics.
- Currently there are attempts to include gravity.
  - String theory.

The Modeling Process

- It is based on experiment and/or observation.
- It is iterative.
  - For motion we have $\geq 6$ iterations.
  - After each change in the model it must be checked by further experimentation and observation.
- It is rare that a model captures all aspects of the phenomenon.

Linear Motion

- Motion in one dimension — $x(t)$ is the distance from a reference position.
- Example: motion of a ball in the earth’s gravity — $x(t)$ is the height of the ball above the surface of the earth.
- Velocity: $v = x'$
- Acceleration: $a = v' = x''$. 
• Acceleration due to gravity is (approximately) constant near the surface of the earth
\[ F = -mg, \quad \text{where} \quad g = 9.8 \text{m/s}^2 \]

• Newton’s second law: \( F = ma \)

• Equation of motion: \( ma = -mg \),
which becomes
\[ x'' = -g \quad \text{or} \quad x' = v, \]
\[ v' = -g. \]

• Solving the system \( x' = v, \quad v' = -g \)
• Integrate the second equation:
\[ v(t) = -gt + c_1 \]
• Substitute into the first equation and integrate:
\[ x(t) = -\frac{1}{2}gt^2 + c_1t + c_2. \]

Resistance of the Medium

• Force of resistance
\[ R(x, v) = -r(x, v)v \quad \text{where} \quad r(x, v) \geq 0. \]

• Resistance proportional to velocity.
\[ R(x, v) = -rv, \quad r \text{ a positive constant.} \]

• Magnitude of resistance proportional to the square of the velocity.
\[ R(x, v) = -k|v|v, \quad k \text{ a positive constant.} \]
\[ R(x, v) = -rv \]

- Total force: \( F = -mg - rv \)
- Newton's second law: \( F = ma \)
- Equation of motion:
  \[ mx'' = -mg - rv \quad \text{or} \quad x' = v, \quad v' = -\frac{mg + rv}{m} \]

\[ R(x, v) = -k|v|v \]

- Total force: \( F = -mg - k|v|v \).
- Equation of motion:
  \[ mx'' = -mg - k|v|v \quad \text{or} \quad x' = v, \quad v' = -g - \frac{k|v|v}{m}. \]
- The equation for \( v \) is separable.

- The equation \( v' = -\frac{mg + rv}{m} \) for \( v \) is separable.
- Solution is \( v(t) = Ce^{-rt/m} - \frac{mg}{r} \).
- Notice
  \[ \lim_{t \to \infty} v(t) = -\frac{mg}{r} \]
- The terminal velocity is \( v_{\text{term}} = -\frac{mg}{r} \).
Suppose a ball is dropped from a high point. Then $v < 0$.

The equation is

$$v' = -\frac{mg}{m} + kv^2$$

$$= -\frac{k}{m} (mg - v^2)$$

$$= -\frac{k}{m} (\alpha^2 - v^2)$$

where $\alpha = \sqrt{mg/k}$.

The solution is

$$v(t) = \sqrt{\frac{mg}{k}} A e^{-\frac{2\alpha}{k} \sqrt{mg/m} t} - 1.$$  

The terminal velocity is

$$v_{\text{term}} = -\sqrt{mg/k}.$$  

Solving for $x(t)$

Integrating $x' = v(t)$ is sometimes hard.

Use the trick (see Exercise 2.3.7):

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

The equation

$$v \frac{dv}{dx} = a$$

is usually separable.
A ball is projected from the surface of the earth with velocity $v_0$. How high does it go?

- At $t = 0$, we have $x(0) = 0$ and $v(0) = v_0$.
- At the top we have $t = T$, $x(T) = x_{\text{max}}$, and $v(T) = 0$.
- $R = 0 \Rightarrow a = -g$.

\[
v dv = -g \, dx
\]
\[
\int_0^v v \, dv = - \int_0^{x_{\text{max}}} g \, dx
\]
\[
x_{\text{max}} = \frac{v_0^2}{2g}
\]

**Problem Trick 23**

- $R = -rv \Rightarrow a = -g - rv/m$.

\[
\frac{v \, dv}{v + mg/r} = - \frac{r}{m} \, dx.
\]
\[
x_{\text{max}} = \frac{m}{r} \left[ \frac{mg}{r} \ln \left( 1 + \frac{v_0r}{mg} \right) - v_0 \right].
\]

**Problem Trick 24**

- $R = -kv \Rightarrow a = -g - kv^2/m$.

\[
\frac{v \, dv}{v^2 + mg/k} = - \frac{k}{m} \, dx.
\]
\[
x_{\text{max}} = \frac{m}{2k} \ln \left( 1 + \frac{kv_0^2}{mg} \right).
\]