Math 211 Sec. 4

Linear Equations

Mixing Problems

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A linear equation is one of the form

\[ x' = a(t)x + f(t). \]

- Example: \( x' = \tan(t)x + 3\sin^2(t) \)
- The unknown function \( x \) and its derivative must appear \textit{linearly}.
- The equation is \textit{homogeneous} if \( f = 0 \)
  - \( x' = a(t)x \), e.g. \( x' = \tan(t)x \)
- The equation is \textit{inhomogeneous} if \( f \neq 0 \)
Homogenous Linear Equations

- Homogeneous linear equations are separable.

\[ \frac{dx}{dt} = a(t)x \]

\[ \frac{dx}{x} = a(t) \, dt \]

\[ \ln |x(t)| = \int a(t) \, dt \]

\[ x(t) = Ae^{\int a(t) \, dt} \]
Example: \( x' = \tan(t)x \).

\[
x(t) = Ae\int \tan(t) \, dt
\]
\[
= Ae^{-\ln(\cos(t))}
\]
\[
= A\cos t
\]
\[
= A\sec t
\]
Example: $x' = \tan(t)x + 3\sin^2(t)$

- Rewrite as $x' - \tan(t)x = 3\sin^2(t)$
- Multiply by $\cos t$.

$$\cos(t)x' - \sin(t)x = 3\sin^2(t)\cos(t)$$

The left hand side is the derivative of $\cos(t)x$. So

$$[\cos(t)x]' = 3\sin^2(t)\cos(t)$$
• Integrate

\[ \cos(t)x(t) = 3 \int \sin^2(t) \cos(t) \, dt = \sin^3(t) + C \]

• Solve for \( x \)

\[ x(t) = \frac{\sin^3(t) + C}{\cos(t)}. \]

How did we do that? Can we do it in general?
The Key Step for $x' = ax + f$

- Rewrite as $x' - ax = f$.
- Multiply by a function $u(t)$ so that

$$u[x' - ax] = [ux]'$$

$$ux' - aux = ux' + u'x$$

- True if $u' = -au$. Linear, homogeneous

$$u(t) = e^{-\int a(t) \, dt}$$ is one solution.

- $u$ is called an integrating factor.
Solving the Linear Equation

\[ x' = a(t)x + f(t) \]

Four step process:

1. **Rewrite** as \( x' - ax = f \).

2. **Multiply** by the **integrating factor**
   \[
   u(t) = e^{-\int a(t) \, dt}.
   \]

   Equation becomes \([ux]' = ux' - aux = uf\).

3. **Integrate**:
   \[ u(t)x(t) = \int u(t)f(t) \, dt + C. \]

4. **Solve** for \( x(t) \).
Examples

• $x' = -4x + 8$, $x(0) = 0$.

• $x' = 2tx + e^{t^2}$, $x(0) = 1$.

• $y' = 3y - t$, $y(0) = 2$.

• $z' = (z + 1) \cos t$, $z(\pi) = -1$. 

Solution method
Mixing Problem #1

A tank originally holds 500 gallons of pure water. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 5 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.
Model

- \( S(t) = \) the amount of sugar in the tank in lbs.
- *Concentration* = pounds per unit volume.
  - \( c(t) = \frac{S(t)}{V} \text{ lbs gal} \).
- Modeling is easier in terms of the total amount, \( S(t) \).
- Draw a picture.
Balance Law

- Rate of change = Rate in - Rate out
- Rate = volume rate × concentration
- For the problem

\[ \text{Rate in} = 5 \frac{\text{gal}}{\text{min}} \times \frac{1}{2} \frac{\text{lb}}{\text{gal}} = 2.5 \frac{\text{lb}}{\text{min}} \]

\[ \text{Rate out} = 5 \frac{\text{gal}}{\text{min}} \times \frac{S}{500} \frac{\text{lb}}{\text{gal}} = \frac{S}{100} \frac{\text{lb}}{\text{min}} \]
Solution

\[
\frac{dS}{dt} = \text{Rate in} - \text{Rate out}
\]

\[= 2.5 - \frac{S}{100}.\]

- General solution: \(S(t) = 250 + Ce^{-t/100}.\)
- Particular solution: \(S(t) = 250(1 - e^{-t/100}).\)
Other possible initial conditions

- There is initially 20 lbs of sugar in the tank.
- The concentration of sugar in the tank at $t = 0$ is 1 lb/gallon.
Mixing Problem #2

A tank originally holds 500 gallons of sugar water with a concentration of \( \frac{1}{10} \) lb/gal. At \( t = 0 \) there starts a flow of sugar water into the tank with a concentration of \( \frac{1}{2} \) lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 10 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.
Solution

• Rate in = $5 \frac{\text{gal}}{\text{min}} \times \frac{1}{2} \frac{\text{lb}}{\text{gal}} = 2.5 \frac{\text{lb}}{\text{min}}$

• Rate out = $10 \frac{\text{gal}}{\text{min}} \times \frac{S}{V} \frac{\text{lb}}{\text{gal}}$

♦ $V(t) = 500 - 5t$.

♦ Rate out = $\frac{10S}{500 - 5t} \frac{\text{lb}}{\text{min}}$
\[
\frac{dS}{dt} = \text{Rate in} - \text{Rate out} = 2.5 - \frac{2S}{100 - t},
\]

- **General solution:**
  \[
  S(t) = 2.5(100 - t) + C(100 - t)^2.
  \]

- **Particular solution:**
  \[
  S(t) = 2.5(100 - t) - \frac{(100 - t)^2}{50}.
  \]