Math 211

Lecture #3

Separable Equations

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Autonomous Equations

General equation:
\[ \frac{dy}{dt} = f(t, y) \]
\[ \frac{dy}{dt} = t - y^2 \]

Autonomous equation:
\[ \frac{dy}{dt} = f(y) \]
\[ \frac{dy}{dt} = y(2 - y)/3 \]

In an *autonomous equation* the right-hand side has no explicit dependence on the independent variable.
Equilibrium Points

Autonomous equation:

\[
\frac{dy}{dt} = f(y) \quad \frac{dy}{dt} = \frac{y(2 - y)}{3}
\]

- **Equilibrium point:**

  \[f(y_0) = 0 \quad y_0 = 0 \quad \text{or} \quad 2\]

- **Equilibrium solution:**

  \[y(t) = y_0 \quad y(t) = 0 \quad \text{and} \quad y(t) = 2\]
Between Equilibrium Points

- \( \frac{dy}{dt} = f(y) > 0 \implies y(t) \text{ is increasing.} \)
- \( \frac{dy}{dt} = f(y) < 0 \implies y(t) \text{ is decreasing.} \)

Example: \( \frac{dy}{dt} = \frac{y(2 - y)}{3} \)
Separable Equations

General differential equation:
\[
\frac{dy}{dt} = f(t, y) \quad \frac{dy}{dt} = t - y^2
\]

Separable differential equation:
\[
\frac{dy}{dt} = g(y)h(t) \quad \frac{dy}{dt} = t \sec y
\]

In a *separable equation* the right-hand side is a product of a function of the independent variable \((t)\) and a function of the unknown function \((y)\).

- Autonomous equations are separable.
Solving Separable Equations

\[
\frac{dy}{dt} = t \sec y
\]

- Step 1: Separate the variables:

\[
\frac{dy}{\sec y} = t \, dt \quad \text{or} \quad \cos y \, dy = t \, dt
\]

We have to worry about dividing by 0, but in this case \( \sec y \) is never equal to 0.
Step 2: Integrate both sides

\[
\int \cos y \, dy = \int t \, dt
\]

\[
\sin(y) + C_1 = \frac{1}{2} t^2 + C_2 \quad \text{or}
\]

\[
\sin(y) = \frac{1}{2} t^2 + C
\]

where \( C = C_2 - C_1 \).
Step 3: Solve for $y(t)$

$$\sin(y) = \frac{1}{2}t^2 + C$$

$$y(t) = \arcsin \left( C + \frac{1}{2}t^2 \right).$$

This is the general solution to $\frac{dy}{dt} = t \sec y$. 
Solving Separable Equations

\[ \frac{dy}{dt} = g(y)h(t) \]

The three step solution process:

1. **Separate** the variables. \[ \frac{dy}{g(y)} = h(t) \, dt \]

2. **Integrate** both sides. \[ \int \frac{dy}{g(y)} = \int h(t) \, dt \]

3. **Solve** for \( y(t) \).
Examples

- $y' = ry$
- $y' = 2ty$
- $R' = \frac{\sin t}{1 + R}$ with $R(0) = 1, -2, -1$
- $x' = \frac{3t^2x}{1 + 2x^2}$ with $x(0) = 1, 0$
- $y' = 1 + y^2$ with $y(0) = -1, 0, 1$
Why the Method Works

\[
\frac{dy}{dt} = g(y) h(t)
\]

\[
\frac{1}{g(y)} \frac{dy}{dt} = h(t) \quad \text{if } g(y) \neq 0
\]

\[
\int \frac{1}{g(y)} \frac{dy}{dt} \, dt = \int h(t) \, dt
\]

\[
\int \frac{1}{g(y)} \, dy = \int h(t) \, dt
\]
Models of Motion

History of models of planetary motion

- Babylonians - 3000 years ago
  - Initiated the systematic study of astronomy.
Greeks

- Descriptive model
  - Geocentric model
  - Epicycles
- Enabled predictions
- No causal explanation
Nicholas Copernicus (1543)

- Shifted the center of the universe to the sun.
- Less epicycles required.
- Still descriptive and not causal.
- Major change in human understanding of their place in the universe.
Johann Kepler (1609)

- Based on experimental work of Tycho Brahe.
- Ellipses instead of epicycles.
  - Sun at a focus of the ellipse.
- Three laws of planetary motion.
- Still descriptive and not causal.
Isaac Newton

- Three major contributions.
  - Fundamental theorem of calculus.
    - Invention of calculus.
  - Laws of mechanics.
    - Second law — \( F = ma \).
  - Universal law of gravity.
  - *Principia Mathematica* 1687
Isaac Newton

- Laws of mechanics and gravitation were based on his own experiments and his understanding of the experiments of others.
- Derived Kepler’s three laws of planetary motion.
- Causal explanation.
  - For any mechanical motion.
Isaac Newton


- Problems
  - Force of gravity was action at a distance.
  - Physical anomalies.
Albert Einstein

- Special theory of relativity – 1905.
- General theory of relativity – 1916.
  - Gravity is due to curvature of space-time.
  - Curvature is caused by mass.
  - Explains action at a distance.
- All known anomalies explained.
Unified Theories

- Four fundamental forces.
  - Gravity, electromagnetism, strong nuclear, and weak nuclear.
- Last three unified by quantum mechanics.
  - Quantum chromodynamics.
- Attempts to include gravity.
  - String theory.
Unified Theories

- String theory.
The Modeling Process

• It is based on experiment and/or observation.

• It is iterative.
  ♦ For motion we have \( \geq 6 \) iterations.
  ♦ After each change in the model it must be checked by experimentation and observation.

• It is rare that a model captures all aspects of the phenomenon.