Math 211

Lecture #10

Financial Models

September 19, 2001

Compound Interest

• Put some money into an account that returns a percentage each year, compounded continuously. How will it grow?
  • "Some money" is \( P_0 \) measured in $1000.
  • "Returns a percentage" is \( r \)%/year.
  • "Some time later" is measured in years.
  • "Compounded continuously" \( \Rightarrow \) \( P' = rP \).

Solution

\[ P(t) = P_0 e^{rt} \]

• The principal grows exponentially.
• If \( r = 8\% \), then after 20 years
  \[ P(20) = P_0 e^{0.08 \times 20} \]
  \[ = 4.953P_0 \]
• After 40 years \( P(40) = 24.5325P_0 \).
Retirement Account

• Set up a retirement account by investing an initial amount. In addition, deposit a fixed amount each year until you retire. Assume it returns a percentage each year, compounded continuously. How much is there some time later?

• “A fixed amount each year” is \( D \), measured in $1,000 each year. We assume this is invested continuously.

Retirement Account

• The model is

\[ P' = rP + D. \]

• Solution

\[ P(t) = P_0 e^{rt} + \frac{D}{r} \left( e^{rt} - 1 \right). \]

Retirement Account

• Suppose you start with an investment of $1,000 at the age of 25, and invest $100 each month until you retire at 65. The account returns 8% per year. How much is in the retirement account when you retire?

• \( P_0 = 1000 \), \( D = 100 \times 12 = 1200 \), \( r = 8\% = 0.08 \).

• At 65 the principal is $377,521.

• Is this enough to retire on?
Retirement Planning

- If you need a certain income after you retire, how much must you have in your retirement account when you retire?
- “Certain income” is $I$ (in $1000/year) withdrawn from the account.
- “How much” is the amount $P_0$ in the account at retirement.
- The account still grows due to its return at $r\%$/year.

Return Definitions

- The model is
  \[ P' = rP - I, \quad P(0) = P_0. \]
- Solution $P(t) = P_0 e^{rt} - \frac{I}{r} [e^{rt} - 1]$.
- We are given $I$, $r$, & $P(yd)$.
- We need to compute $P_0$.

Example

- If you will need an income of $75,000 for 30 years after retirement and your account returns 6%, your account balance at retirement should be $1,043,000.
Retirement Planning

- Instead of investing a fixed amount each month, it would be more realistic to invest a percentage of your salary. What should this percentage be in order to accumulate an adequate investment balance? Include the effect of inflation.
- You starting salary is $S_0$.
- Assume it will increase at $s\%$ per year.
  - Then $S'(t) = sS(t)$, or $S(t) = S_0e^{st}$.

Retirement Planning

- The model for the growth of the retirement account is
  $$P' = rP + \lambda S_0e^{st} \quad \text{with} \quad P(0) = P_0.$$  
- Solution
  $$P(t) = P_0e^{rt} + \frac{\lambda S_0}{r - s} [e^{st} - e^{rt}].$$

Retirement Planning

- Assume
  - $P_0 = $1,000 and $r = 8\%$
  - $S_0 = $35,000 and $s = 4\%$
    - Notice that $S(40) = $173,356.
  - Need a retirement income of $150,000.
  - Aim for a balance at retirement of $2,000,000.
  - Requires $\lambda = 11.53\%$.  

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Other Strategies

- Delayed gratification. Deposit a percentage of your salary that starts at \( \lambda \% \), and decays linearly to 0 over 40 years.

\[
P' = rP + \lambda (1 - t/40)S_0 e^{rt}
\]

- Immediate gratification. Deposit a percentage of your salary that starts at 0 and grow linearly over 40 years to \( \lambda \% \).

\[
P' = rP + \frac{\lambda}{40} S_0 e^{rt}
\]