Compound Interest

- Put some money into an account that returns a percentage each year, compounded continuously. How will it grow?

  - “Some money” is $P_0$ measured in $1000$.
  - “Returns a percentage” is $r\%$/year.
  - “Some time later” is measured in years.
  - “Compounded continuously” $\Rightarrow P' = rP$. 
## Compound Interest

- **Solution**

\[ P(t) = P_0e^{rt} \]

- The principal grows exponentially.

- If \( r = 8\% \), then after 20 years

\[ P(20) = P_0e^{0.08 \times 20} \]

\[ = 4.953 \, P_0 \]

- After 40 years \( P(40) = 24.5325 \, P_0 \).
Retirement Account

Set up a retirement account by investing an initial amount. In addition, deposit a fixed amount each year until you retire. Assume it returns a percentage each year, compounded continuously. How much is there some time later?

“A fixed amount each year” is $D$, measured in $1,000 each year. We assume this is invested continuously.
Retirement Account

- The model is

\[ P' = rP + D. \]

- Solution

\[ P(t) = P_0e^{rt} + \frac{D}{r}[e^{rt} - 1]. \]
Suppose you start with an investment of $1,000 at the age of 25, and invest $100 each month until you retire at 65. The account returns 8% per year. How much is in the retirement account when you retire?

- $P_0 = 1000, \ D = 100 \times 12 = 1200, \ r = 8\% = 0.08.$

- At 65 the principal is $377,521.

- Is this enough to retire on?
Retirement Planning

- If you need a certain income after you retire, how much must you have in your retirement account when you retire?
  - “Certain income” is $I$ (in $1000/year) withdrawn from the account.
  - “How much” is the amount $P_0$ in the account at retirement.
  - The account still grows due to its return at $r\%$/year.
Retirement Planning

- The model is

\[ P' = rP - I, \quad P(0) = P_0. \]

- Solution \( P(t) = P_0 e^{rt} - \frac{I}{r} [e^{rt} - 1]. \)

- We are given \( I, r, \) & \( P(y_d). \)

- We need to compute \( P_0. \)
Retirement Planning

- If you will need an income of $75,000 for 30 years after retirement and your account returns 6%, your account balance at retirement should be

  $1,043,000.
Retirement Planning

• Instead of investing a fixed amount each month, it would be more realistic to invest a percentage of your salary. What should this percentage be in order to accumulate an adequate investment balance? Include the effect of inflation.

• You starting salary is \( S_0 \).

• Assume it will increase at \( s\% \) per year.

  ♦ Then \( S' = sS \), or \( S(t) = S_0 e^{st} \).
Retirement Planning

- The model for the growth of the retirement account is

\[ P' = rP + \lambda S_0 e^{st} \quad \text{with} \quad P(0) = P_0. \]

- Solution

\[ P(t) = P_0 e^{rt} + \frac{\lambda S_0}{r - s} \left[ e^{rt} - e^{st} \right]. \]
Retirement Planning

- Assume
  - $P_0 = $1,000 and $r = 8\%$
  - $S_0 = $35,000 and $s = 4\%$
    - Notice that $S(40) = $173,356.
  - Need a retirement income of $150,000.
    - Aim for a balance at retirement of $2,000,000.
- Requires $\lambda = 11.53\%$. 
Other Strategies

- Delayed gratification. Deposit a percentage of your salary that starts at $\lambda\%$, and decays linearly to 0 over 40 years.

\[ P' = rP + \lambda(1 - t/40)S_0 e^{st} \]

- Immediate gratification. Deposit a percentage of your salary that starts at 0 and grow linearly over 40 years to $\lambda\%$.

\[ P' = rP + \frac{\lambda t}{40} S_0 e^{st} \]