Math 211

Lecture #15
Systems of Linear Equations

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Example

Solve

\[\begin{align*}
3x - 4y + 5z &= 3 \\
-x + 2y - 2z &= -2
\end{align*}\]

• Find all solutions.
• Find a systematic method which works for all systems, no matter how large.

Vectors and Matrices

• Introduce the vectors

\[
\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}
\]

and the matrix

\[
\mathbf{C} = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix}
\]

• \(\mathbf{x}\) is the vector of unknowns, \(\mathbf{b}\) is the RHS, and \(\mathbf{C}\) is the coefficient matrix.
• We will define the product \(\mathbf{C}\mathbf{x}\) so that the system can be written as \(\mathbf{C}\mathbf{x} = \mathbf{b}\).
Vectors

- A vector is a list of numbers
- 2-vectors, 3-vectors, n-vectors
- Row vectors and column vectors.
- A vector has length and direction
  - Parallel vectors are equal
  - Transpose of a vector, \( v^T \).

Algebra of Vectors

- Addition of Vectors
  - Algebraic view of addition
  - Geometric view of addition
  - Addition of more than two vectors
- Multiplication by a Scalar
  - Algebraic view
  - Geometric view

Linear Combinations of Vectors

- Vectors \( x = (2, -3)^T \) and \( y = (-1, 2)^T \).
- Any vector of the form \( a x + b y \) is a linear combination of \( x \) and \( y \).
- \( 3x + 2y = (4, -5)^T \).
- Any 2-vector is a linear combination of \( x \) and \( y \).
- Linear combinations of more than two vectors.
Matrices

- A matrix is a rectangular array of numbers.
- Example
  \[
  A = \begin{pmatrix}
  -1 & 0 & 2 & 6 \\
  0 & 3 & -4 & 10 \\
  3 & 3 & 2 & -5 \\
  \end{pmatrix}
  \]
- Size of \( A \) = (3,4); 3 rows & 4 columns.
  - 3 row vectors and 4 column vectors.

Linear Combinations and Systems

- The example system can be written as a vector equation
  \[
  \begin{pmatrix}
  3x - 4y + 5z \\
  -x + 2y - 2z 
  \end{pmatrix}
  = \begin{pmatrix}
  3 \\
  -2 
  \end{pmatrix}
  \]
- or
  \[
  x \begin{pmatrix}
  3 \\
  -1 
  \end{pmatrix} + y \begin{pmatrix}
  -4 \\
  2 
  \end{pmatrix} + z \begin{pmatrix}
  5 \\
  -2 
  \end{pmatrix} = \begin{pmatrix}
  3 \\
  -2 
  \end{pmatrix}
  \]
- These vectors are the column vectors in the coefficient matrix.

Coefficient Matrix

- The coefficient matrix is
  \[
  C = \begin{pmatrix}
  3 & -4 & 5 \\
  -1 & 2 & -2 
  \end{pmatrix}
  \]
- Solving the system of equations \( \iff \) finding a linear combination of the columns of the coefficient matrix which is equal to the RHS.
Product of a Matrix with a Vector

- The product of a matrix $A$ and a vector $x$ is the linear combination of the columns of $A$ with the elements of $x$ as coefficients.
- Example:

\[
\begin{pmatrix}
3 & -4 & 5 \\
-1 & 2 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

\[
= x \begin{pmatrix} 3 \\ -1 \end{pmatrix} + y \begin{pmatrix} -4 \\ 2 \end{pmatrix} + z \begin{pmatrix} 5 \\ -2 \end{pmatrix}
\]

Example

- Thus the system of equations becomes

\[
\begin{pmatrix}
3 & -4 & 5 \\
-1 & 2 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix} 3 \\ -2 \end{pmatrix}
\]

or

\[
C'x = b
\]

Computing the Product of a Matrix and a Vector.

- From the definition.
- A faster way.
  - $A = (a_{ij})$, a $p \times q$ matrix, and $x$, a column $q$-vector.

\[
Ax = y \iff \quad y_i = \sum_{j=1}^q a_{ij}x_j \quad \text{for } 1 \leq i \leq p.
\]

- $Ax$ is only defined if $A$ has the same number of columns as $x$ has rows.
Algebraic Properties of the Matrix-Vector Product

Suppose $A$ is a matrix, $x$ and $y$ are vectors, and $a$ and $b$ are numbers.

- $A(ax) = a(Ax)$
- $A(x + y) = Ax + Ay$
- $A(ax + by) = aAx + bAy$
- Multiplication by a matrix is a linear operation.

Product of Two Matrices

Suppose $A$ is $n \times p$ and $B$ is $p \times q$.
Write $B$ in terms of its column vectors
\[ B = [b_1 \ b_2 \ldots \ b_q] \]

Define the product $AB$ by
\[ AB = [Ab_1 \ Ab_2 \ldots \ Ab_q] \]

Algebraic Properties of the Product

Suppose that $A$, $B$, and $C$ are matrices

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- However $AB \neq BA$ in general
The Identity Matrix

- In dimension 3
  
  $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- $Ix = x$ for every 3-vector $x$.
- $IA = A$ for every matrix $A$ with 3 rows.
- $AI = A$ for every matrix $A$ with 3 columns.