Math 211

Lecture #15

Systems of Linear Equations

October 1, 2001
Example

Solve

\[3x - 4y + 5z = 3\]
\[-x + 2y - 2z = -2\]

- Find all solutions.
- Find a systematic method which works for all systems, no matter how large.
Vectors and Matrices

- Introduce the vectors

\[ \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \]

and the matrix

\[ \mathbf{C} = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \]

- \( \mathbf{x} \) is the vector of unknowns, \( \mathbf{b} \) is the RHS, and \( \mathbf{C} \) is the coefficient matrix.

- We will define the product \( \mathbf{C}\mathbf{x} \) so that the system can be written as \( \mathbf{C}\mathbf{x} = \mathbf{b} \).
Vectors

- A vector is a list of numbers
- 2-vectors, 3-vectors, $n$-vectors
- Row vectors and column vectors.
- A vector has length and direction
  - Parallel vectors are equal
- Transpose of a vector, $\mathbf{v}^T$. 
Algebra of Vectors

- Addition of Vectors
  - Algebraic view of addition
  - Geometric view of addition
  - Addition of more than two vectors

- Multiplication by a Scalar
  - Algebraic view
  - Geometric view
Linear Combinations of Vectors

- Vectors $\mathbf{x} = (2, -3)^T$ and $\mathbf{y} = (-1, 2)^T$.
- Any vector of the form $a\mathbf{x} + b\mathbf{y}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$.
- $3\mathbf{x} + 2\mathbf{y} = (4, -5)^T$.
- Any 2-vector is a linear combination of $\mathbf{x}$ and $\mathbf{y}$.
- Linear combinations of more than two vectors.
Matrices

- A matrix is a rectangular array of numbers.
- Example

\[
A = \begin{pmatrix}
-1 & 0 & 2 & 6 \\
0 & 3 & -4 & 10 \\
3 & 3 & 2 & -5 \\
\end{pmatrix}
\]

- Size of \( A = (3,4) \); 3 rows & 4 columns.
  - 3 row vectors and 4 column vectors.
Linear Combinations and Systems

- The example system can be written as a vector equation

\[
\begin{pmatrix}
3x - 4y + 5z \\
-x + 2y - 2z
\end{pmatrix}
= \begin{pmatrix} 3 \\ -2 \end{pmatrix}
\]

- or

\[
x \begin{pmatrix} 3 \\ -1 \end{pmatrix} + y \begin{pmatrix} -4 \\ 2 \end{pmatrix} + z \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}
\]

- These vectors are the column vectors in the coefficient matrix.
Coefficient Matrix

- The coefficient matrix is

\[ C = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \]

- Solving the system of equations $\iff$ finding a linear combination of the columns of the coefficient matrix which is equal to the RHS.
Product of a Matrix with a Vector

- The *product* of a matrix $A$ and a vector $x$ is the linear combination of the columns of $A$ with the elements of $x$ as coefficients.

- Example:

$$
\begin{pmatrix}
3 & -4 & 5 \\
-1 & 2 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= x \begin{pmatrix}
3 \\
-1
\end{pmatrix} + y \begin{pmatrix}
-4 \\
2
\end{pmatrix} + z \begin{pmatrix}
5 \\
-2
\end{pmatrix}
$$
Example

• Thus the system of equations becomes

\[
\begin{pmatrix}
3 & -4 & 5 \\
-1 & 2 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
3 \\
-2
\end{pmatrix}
\]

or

\[Cx = b\]
Computing the Product of a Matrix and a Vector.

- From the definition.
- A faster way.
  - \( A = (a_{ij}) \), a \( p \times q \) matrix, and \( x \), a column \( q \)-vector.

\[
Ax = y \iff y_i = \sum_{j=1}^{q} a_{ij}x_j \quad \text{for} \quad 1 \leq i \leq p.
\]

- \( Ax \) is only defined if \( A \) has the same number of columns as \( x \) has rows.
Algebraic Properties of the Matrix-Vector Product

Suppose $A$ is a matrix, $x$ and $y$ are vectors, and $a$ and $b$ are numbers.

- $A(ax) = a(Ax)$
- $A(x + y) = Ax + Ay$
- $A(ax + by) = aAx + bAy$
- Multiplication by a matrix is a linear operation.
Product of Two Matrices

Suppose $A$ is $n \times p$ and $B$ is $p \times q$.

Write $B$ in terms of its column vectors

$$B = [b_1 \ b_2 \ \ldots \ b_q]$$

Define the *product* $AB$ by

$$AB = [Ab_1 \ Ab_2 \ \ldots \ Ab_q]$$
Algebraic Properties of the Product

Suppose that $A$, $B$, and $C$ are matrices

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- However $AB \neq BA$ in general
The Identity Matrix

• In dimension 3

\[ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

• \(Ix = x\) for every 3-vector x.

• \(IA = A\) for every matrix A with 3 rows.

• \(AI = A\) for every matrix A with 3 columns.