Method of Solution for $Ax = b$

- Use the augmented matrix $M = [A, b]$.
- Eliminate as many coefficients as possible.
- Use row operations to reduce to row echelon form.
- Write down the simplified system.
- Backsolve.
  - Assign arbitrary values to the free variables.
  - Solve for the pivot variables.

Consistent Systems

- A system is consistent if it has solutions.
  - The solution set is not the empty set.
- A system is consistent if and only if the simplified system is consistent.
- This is true if and only if the last column (after elimination) does not contain a pivot.
Examples

\[ A = \begin{pmatrix} -3 & 6 & 0 \\ -2 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -9 \\ -6 \\ 7 \end{pmatrix} \]

- Use \( A_1, b_1, \) & \( b_2 \)

Homogeneous Systems

Example \( Ax = 0 \).

\[ A = \begin{pmatrix} -5 & -4 & -2 \\ -5 & -6 & -2 \\ 30 & 27 & 11 \end{pmatrix} \Rightarrow \begin{pmatrix} -5 & -4 & -2 & 0 \\ -5 & -6 & -2 & 0 \\ 30 & 27 & 11 & 0 \end{pmatrix} \]

- Use \( A_2 \)
- During elimination the column of zeros is unchanged.
- It is unnecessary to augment a homogeneous system.

Square Matrices

- There are special kinds:
  - Singular and nonsingular.
  - Invertible and noninvertible.
- What do the terms mean?
- What are the relations between them?
Singular and Nonsingular Matrices

The $n \times n$ matrix $A$ is nonsingular if the equation $Ax = b$ has a solution for any right hand side $b$.

Proposition: The $n \times n$ matrix $A$ is nonsingular if and only if the simplified matrix has only nonzero entries along the diagonal.
- In reduced row echelon form we get $I$.

Examples

\[
A = \begin{pmatrix}
-17 & -16 & -6 \\
18 & 18 & 6 \\
6 & 3 & 3
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
-17 & -16 & -6 \\
18 & 18 & 6 \\
6 & 3 & 4
\end{pmatrix}
\]
- Use $A3$

Proposition: If the $n \times n$ matrix $A$ is nonsingular then the equation $Ax = b$ has a unique solution for any right hand side $b$.

Proposition: The $n \times n$ matrix $A$ is singular if and only if the homogeneous equation $Ax = 0$ has a non-zero solution.
- This is a result that we will use repeatedly.
Invertible Matrices

An $n \times n$ matrix $A$ is invertible if there is an $n \times n$ matrix $B$ such that $AB = BA = I$. The matrix $B$ is called an inverse of $A$.

- If $B_1$ and $B_2$ are both inverses of $A$, then
  
  \[ B_1 = B_1(AB_2) = (B_1A)B_2 = B_2 \]

- The inverse of $A$ is denoted by $A^{-1}$.
- Invertible $\Rightarrow$ nonsingular.

Computing the inverse $A^{-1}$

- Form the matrix $[A, I]$.
- Do elimination until the matrix has the form $[I, B]$.
- Then $A^{-1} = B$.
- A matrix is invertible if and only if it is nonsingular.
- Example $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- Use A3

Structure of the Solution Set

Theorem: Let $x_p$ be a particular solution to $Ax_p = b$.

1. If $Ax_h = 0$ then $x = x_p + x_h$ also satisfies $Ax = b$.
2. If $Ax = b$, then there is a vector $x_h$ such that $Ax_h = 0$ and $x = x_p + x_h$.

- Solution set for $Ax = b$ is known if we know one particular solution $x_p$ and the solution set for the homogeneous system $Ax_h = 0$. 
Solution Set of a Homogeneous System

Our goal is to understand such sets better. In particular we want to know:

• What are the properties of these solution sets?
• Is there a convenient way to describe them?

Nullspace of a Matrix

The nullspace of a matrix \( A \) is the set
\[
\{ x \mid Ax = 0 \}.
\]

• The nullspace of \( A \) is the same as the solution set for the homogeneous system \( Ax = 0 \).
• The nullspace of \( A \) is denoted by \( \text{null}(A) \).

Properties of the Nullspace of \( A \)

Proposition: Let \( A \) be a matrix.

1. If \( x \) and \( y \) are in \( \text{null}(A) \), then \( x + y \) is in \( \text{null}(A) \).
2. If \( a \) is a scalar and \( x \) is in \( \text{null}(A) \), then \( ax \) is in \( \text{null}(A) \).
• \( \text{null}(A) \) has some of the same properties as \( \mathbb{R}^n \).
**Subspaces of \( \mathbb{R}^n \)**

**Definition:** A nonempty subset \( V \) of \( \mathbb{R}^n \) that has the properties

1. if \( x \) and \( y \) are vectors in \( V \), \( x + y \) is in \( V \),
2. if \( a \) is a scalar, and \( x \) is in \( V \), then \( ax \) is in \( V \),

is called a *subspace* of \( \mathbb{R}^n \).

- The nullspace of a matrix is a subspace.

**Examples of Subspaces**

- The nullspace of a matrix is a subspace.
- A line through the origin is a subspace.
  \[ V = \{ tv \mid t \in \mathbb{R} \} \].
- A plane through the origin is a subspace.
  \[ V = \{ av + bw \mid a, b \in \mathbb{R} \} \].
- \( \{0\} \) and \( \mathbb{R}^n \) are subspaces of \( \mathbb{R}^n \).
- These are the *trivial* subspaces.

**Linear Combinations**

**Proposition:** Any linear combination of vectors in a subspace \( V \) is also in \( V \).

- Subspaces of \( \mathbb{R}^n \) have the same kind of linear structure as \( \mathbb{R}^n \) itself.
- In particular the nullspaces of matrices have the same kind of linear structure as \( \mathbb{R}^n \).
Row operations

The permissable operations on the rows of the augmented matrix are called row operations.

- Add a multiple of one row to another.
- Interchange two rows.
- Multiply a row by a non-zero number.

Row Echelon Form

A matrix is in row echelon form if every pivot lies strictly to the right of those in rows above.

\[
\begin{bmatrix}
P & * & * & * & * & * & * & * & * & * & * \\
0 & P & * & * & * & * & * & * & * & * & * \\
0 & 0 & P & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & P & * & * & * & * & * & * & *
\end{bmatrix}
\]

- \(P\) is a pivot, \(*\) is any number.