Subspaces of $\mathbb{R}^n$

Definition: A nonempty subset $V$ of $\mathbb{R}^n$ that has the properties
1. if $x$ and $y$ are vectors in $V$, $x + y$ is in $V$,
2. if $a$ is a scalar, and $x$ is in $V$, then $ax$ is in $V$,
is called a subspace of $\mathbb{R}^n$.
• The nullspace of a matrix is a subspace.
• We are looking for a good way to describe a subspace.

The Span of a Set of Vectors

In every example we have seen the subspace has been the set of all linear combinations of a few vectors.

Definition: The span of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $v_1$, $v_2$, \ldots, and $v_k$ is denoted by
$$\text{span}(v_1, v_2, \ldots, v_k).$$

Proposition: If $v_1$, $v_2$, \ldots, and $v_k$ are all vectors in $\mathbb{R}^n$, then $V = \text{span}(v_1, v_2, \ldots, v_k)$ is a subspace of $\mathbb{R}^n$. 

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Linear Dependence in 2- & 3-D

We need a condition that will keep unneeded vectors out of a spanning list. We will work toward a general definition.

- Two vectors are linearly dependent if one is a scalar multiple of the other.
- Three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent if one is a linear combination of the other two.
- Example: $\mathbf{v}_1 = (1, 0, 0)^T$, $\mathbf{v}_2 = (0, 1, 0)^T$, and $\mathbf{v}_3 = (1, 2, 0)^T$
  
  \[ \mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2. \]
- Notice that $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = 0$.

Linear Dependence

- Three vectors are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
- Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

Linear Independence

Definition: The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are linearly independent if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,
  
  \[ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \]
  
  \[ \Rightarrow c_1 = c_2 = \cdots = c_k = 0, \]
Linear Independence?

How do we decide if a set of vectors is linearly independent? Are the vectors
\[
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \\
\mathbf{v}_2 &= \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \\
\mathbf{v}_3 &= \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}
\end{align*}
\]
linearly independent?

We look at linear combinations of the vectors
\[
c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}
\]
⇔ \([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]c = \mathbf{0}\) where \(c = (c_1, c_2, c_3)^T\)
⇔ \(c \in \text{null}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\).
- \(c = (-3, 2, 1)^T \in \text{null}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\),
\[
\Rightarrow -3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.
\]
- \(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\) are linearly dependent.

Another Example

Are the vectors
\[
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \\
\mathbf{v}_2 &= \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \\
\mathbf{v}_3 &= \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}
\end{align*}
\]
linearly independent?
- \(\text{null}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \{\mathbf{0}\}\).
- \(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\) are linearly independent.
Proposition: Suppose that $v_1, v_2, \ldots, v_k$ are vectors in $\mathbb{R}^n$. Set $V = [v_1, v_2, \ldots, v_k]$.

1. If $\text{null}(V) = \{0\}$, then $v_1, v_2, \ldots, v_k$ are linearly independent.

2. If $c = (c_1, c_2, \ldots, c_k)^T$ is a nonzero vector in $\text{null}(V)$, then
   
   $$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0,$$

   so the vectors are linearly dependent.

Basis of a Subspace

Definition: A set of vectors $v_1, v_2, \ldots, v_k$ form a basis of a subspace $V$ if

1. $V = \text{span}(v_1, v_2, \ldots, v_k)$
2. $v_1, v_2, \ldots, v_k$ are linearly independent.

Examples of Bases

- The vector $v = (1, -1, 1)^T$ is a basis for $\text{null}(A)$.
- $\text{null}(A)$ is the subspace of $\mathbb{R}^3$ with basis $v$.
- The vectors $v = (1, -1, 1, 0)^T$ and $w = (0, -2, 0, 1)^T$ form a basis for $\text{null}(B)$.
- $\text{null}(B)$ is the subspace of $\mathbb{R}^4$ with basis $\{v, w\}$. 
Basis of a Subspace

Proposition: Let \( V \) be a subspace of \( \mathbb{R}^n \).

1. If \( V \neq \{0\} \), then \( V \) has a basis.
2. Every basis of \( V \) has the same number of elements.

Definition: The dimension of a subspace \( V \) is the number of elements in a basis of \( V \).

Example

Find the nullspace of

\[
A = \begin{pmatrix}
3 & -3 & 1 & -1 \\
-2 & 2 & -1 & 1 \\
1 & -1 & 0 & 0 \\
13 & -13 & 5 & -5
\end{pmatrix}.
\]

- \( \text{null}(A) \) is the subspace of \( \mathbb{R}^4 \) with basis \( (1, 1, 0, 0)^T \) and \( (0, 0, 1, -1)^T \).
- \( \text{null}(A) \) has dimension 2.

Example 1

\[
A = \begin{pmatrix}
4 & 3 & -1 \\
-3 & -2 & 1 \\
1 & 2 & 1
\end{pmatrix}
\]

The nullspace of \( A \) is

\( \text{null}(A) = \{a \mathbf{v} \mid a \in \mathbb{R}\} \),

where \( \mathbf{v} = (1, -1, 1)^T \).
Example 2

\[ B = \begin{pmatrix}
4 & 3 & -1 & 6 \\
-3 & -2 & 1 & -4 \\
1 & 2 & 1 & 4
\end{pmatrix} \]

- \( \text{null}(B) = \{ \alpha v + \beta w | \alpha, \beta \in \mathbb{R} \} \), where
  - \( v = (1, -1, 1, 0)^T \) and \( w = (0, -2, 0, 1)^T \).
- \( \text{null}(B) \) consists of all linear combinations of \( v \) and \( w \).