Math 211

Lecture #20

Bases of a Subspace

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Subspaces of $\mathbb{R}^n$

**Definition:** A nonempty subset $V$ of $\mathbb{R}^n$ that has the properties

1. if $x$ and $y$ are vectors in $V$, $x + y$ is in $V$,

2. if $a$ is a scalar, and $x$ is in $V$, then $ax$ is in $V$,

is called a *subspace* of $\mathbb{R}^n$.

- The nullspace of a matrix is a subspace.
- We are looking for a good way to describe a subspace.
The Span of a Set of Vectors

In every example we have seen the subspace has been the set of all linear combinations of a few vectors.

**Definition:** The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

**Proposition:** If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are all vectors in $\mathbb{R}^n$, then $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a *subspace* of $\mathbb{R}^n$. 
Linear Dependence in 2- & 3-D

We need a condition that will keep unneeded vectors out of a spanning list. We will work toward a general definition.

- Two vectors are *linearly dependent* if one is a scalar multiple of the other.
- Three vectors \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) are *linearly dependent* if one is a linear combination of the other two.
  
  - Example: \( \mathbf{v}_1 = (1, 0, 0)^T, \mathbf{v}_2 = (0, 1, 0)^T, \) and \( \mathbf{v}_3 = (1, 2, 0)^T \)
  
  \[ \mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2. \]

  - Notice that \( \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = 0. \)
Linear Dependence

- **Three vectors** are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
  - Non-trivial means that at least one of the coefficients is not 0.

- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
Linear Independence

**Definition:** The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, and $\mathbf{v}_k$ are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k = 0$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$
Linear Independence?

How do we decide if a set of vectors is linearly independent? Are the vectors

\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix} \]

linearly independent?
We look at linear combinations of the vectors

\[ c_1v_1 + c_2v_2 + c_3v_3 = 0 \]

\[ \Leftrightarrow [v_1, v_2, v_3]c = 0 \text{ where } c = (c_1, c_2, c_3)^T \]

\[ \Leftrightarrow c \in \text{null}(v_1, v_2, v_3). \]

- \( c = (-3, 2, 1)^T \in \text{null}(v_1, v_2, v_3), \)
  \[ \Rightarrow -3v_1 + 2v_2 + v_3 = 0. \]

- \( v_1, v_2, v_3 \) are linearly dependent.
Another Example

Are the vectors

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

linearly independent?

- $\text{null}([v_1, v_2, v_3]) = \{0\}$.
- $v_1, v_2, v_3$ are linearly independent.
**Proposition:** Suppose that $v_1, v_2, \ldots, \text{and } v_k$ are vectors in $\mathbb{R}^n$. Set $V = [v_1, v_2, \ldots, v_k]$.

1. If $\text{null}(V) = \{0\}$, then $v_1, v_2, \ldots, \text{and } v_k$ are linearly independent.

2. If $c = (c_1, c_2, \ldots, c_k)^T$ is a nonzero vector in $\text{null}(V)$, then

   $$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0,$$

   so the vectors are linearly dependent.
**Basis of a Subspace**

**Definition:** A set of vectors $v_1, v_2, \ldots, v_k$ form a *basis* of a subspace $V$ if

1. $V = \text{span}(v_1, v_2, \ldots, v_k)$

2. $v_1, v_2, \ldots, v_k$ are **linearly independent**.
Examples of Bases

- The vector $\mathbf{v} = (1, -1, 1)^T$ is a basis for $\text{null}(A)$.
  - $\text{null}(A)$ is the subspace of $\mathbb{R}^3$ with basis $\mathbf{v}$.

- The vectors $\mathbf{v} = (1, -1, 1, 0)^T$ and $\mathbf{w} = (0, -2, 0, 1)^T$ form a basis for $\text{null}(B)$.
  - $\text{null}(B)$ is the subspace of $\mathbb{R}^4$ with basis $\{\mathbf{v}, \mathbf{w}\}$. 
Basis of a Subspace

**Proposition:** Let $V$ be a subspace of $\mathbb{R}^n$.

1. If $V \neq \{0\}$, then $V$ has a basis.
2. Every basis of $V$ has the same number of elements.

**Definition:** The *dimension* of a subspace $V$ is the number of elements in a basis of $V$. 
Example

Find the nullspace of

$$A = \begin{pmatrix} 3 & -3 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 13 & -13 & 5 & -5 \end{pmatrix}.$$ 

- $\text{null}(A)$ is the subspace of $\mathbb{R}^4$ with basis $(1, 1, 0, 0)^T$ and $(0, 0, 1, -1)^T$.
- $\text{null}(A)$ has dimension 2.
Example 1

\[ A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \]

The nullspace of \( A \) is

\[ \text{null}(A) = \{ av \mid a \in \mathbb{R} \}, \]

where \( v = (1, -1, 1)^T \).
Example 2

\[ B = \begin{pmatrix}
4 & 3 & -1 & 6 \\
-3 & -2 & 1 & -4 \\
1 & 2 & 1 & 4
\end{pmatrix} \]

- \( \text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \mid a, b \in \mathbb{R}\} \), where 
  \( \mathbf{v} = (1, -1, 1, 0)^T \) and \( \mathbf{w} = (0, -2, 0, 1)^T \).
- \( \text{null}(B) \) consists of all linear combinations of \( \mathbf{v} \) and \( \mathbf{w} \).