Systems of Differential Equations

Example: SIR model of the spread of an infectious disease, such as the flu, measles, the common cold, etc.

Assumptions:
- The disease is of short duration and rarely fatal.
- The disease spreads through human contact.
- Recovered individuals are immune.

\[ S' = -aSI \]
\[ I' = aSI - bI \]
\[ R' = bI. \]

MATLAB & pplane5.
General System in 2D

\[ x' = f(t, x, y) \]
\[ y' = g(t, x, y) \]

- Example:
  \[ x' = y \]
  \[ y' = -x \]

- Solution: \(x(t) = \sin t\) and \(y(t) = \cos t\)

- Verify by direct substitution.

General System in Higher D

\[ x_1' = f_1(t, x_1, x_2, \ldots, x_n) \]
\[ x_2' = f_2(t, x_1, x_2, \ldots, x_n) \]
\[ \vdots \]
\[ x_n' = f_n(t, x_1, x_2, \ldots, x_n) \]

- The dimension of a system is the number of unknown functions = the number of equations.
- The SIR model has dimension 3 (or 2).

Vector Notation — 2D

- In 2D set \(u_1(t) = x(t)\) & \(u_2(t) = y(t)\), and
  \[ \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \].

- Then in the example
  \[ x' = y \]
  \[ y' = -x \]
  \[ \iff \mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix} \]
Vector Notation — Planar System

- For the general case use vector notation and set
  \[ \mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} f(t, u_1, u_2) \\ g(t, u_1, u_2) \end{pmatrix}. \]
- Then
  \[
  \begin{align*}
  x' &= f(t, x, y) \\
  y' &= g(t, x, y)
  \end{align*}
  \]
  \[ \implies \mathbf{u}' = \mathbf{F}(t, \mathbf{u}) \]

Vector Notation — General

- In higher dimensions, set
  \[
  \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \\ \vdots \\ f_n(t, x) \end{pmatrix}.
  \]
- The general system can be written
  \[ \mathbf{x}' = \mathbf{f}(t, \mathbf{x}). \]

Vector Notation — SIR Model

For the SIR model set \( x_1 = S, \) \( x_2 = I, \) and \( x_3 = R. \) Then
the system can be written
\[
\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} -ax_1x_2 \\ ax_1x_2 - bx_2 \\ bx_2 \end{pmatrix} = \mathbf{f}(\mathbf{x}).
\]
- This is an autonomous system.
- The RHS has no explicit dependence on \( t. \)
Initial Value Problem

\[ x' = f(t, x) \quad x(t_0) = x_0, \]

- Each component of \( x(t_0) \) must be specified.
- Example:

\[
\begin{align*}
  x' &= y \\
  y' &= -x
\end{align*}
\]

with \( x(0) = 2 \) \quad \text{and} \quad y(0) = 13

- SIR model: The initial populations in each category must be specified.

Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

Example of Reduction

- Third-order equation: \( y''' + 2yy' = 3 \cos t \)
- Set \( x_1 = y, x_2 = y', \) and \( x_3 = y'''. \)
- Then

\[
\begin{align*}
  x_1' &= x_2 \\
  x_2' &= x_3 \\
  x_3' &= 3 \cos t - 2x_1x_2
\end{align*}
\]

- This system is not autonomous.
Geometric Interpretation of Solutions

- Parametric plot
- Tangent vectors
- Vector fields
- Phase plane
- pplane5 for planar autonomous systems.