Math 211

Lecture #23
Qualitative Analysis

October 22, 2001

General System in 2D

\[ x' = f(t, x, y) \]
\[ y' = g(t, x, y) \]

• Example:

\[ x' = y \]
\[ y' = -x \]

General System in Higher D

\[ x'_1 = f_1(t, x_1, x_2, \ldots, x_n) \]
\[ x'_2 = f_2(t, x_1, x_2, \ldots, x_n) \]

\[ \vdots \]

\[ x'_n = f_n(t, x_1, x_2, \ldots, x_n) \]
Vector Notation — General

- Set
\[ x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad f(t, x) = \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \\ \vdots \\ f_n(t, x) \end{pmatrix}. \]

- The general system can be written
\[ x' = f(t, x). \]

Initial Value Problem

\[ x' = f(t, x) \quad x(t_0) = x_0. \]

- Each component of \( x(t_0) \) must be specified.
- Example
\[ x' = y \quad y' = -x \quad \text{with} \quad x(0) = 2, \quad y(0) = 13. \]

Existence & Uniqueness

General System \( x' = f(t, x) \)

- \( x \) in an open set \( U \subset \mathbb{R}^n \)
- \( t \) in an interval \( I = (a, b) \)

\[ R = I \times U = \{ (t, x) \mid t \in I \text{ and } x \in U \}. \]
Theorem: Suppose that $f(t,x)$ is continuous in $R$, and that all first partials of $f$ are also continuous in $R$. Then given any $t_0 \in I$ and $x_0 \in U$ there is a unique solution to the initial value problem

$$x' = f(t,x) \quad \text{with} \quad x(t_0) = x_0,$$

defined on an interval containing $t_0$. The solution exists at least until the solution curve $t \rightarrow (t, x(t))$ leaves $R$.

Autonomous Systems

System of the form

$$x' = f(x).$$

- Look at solution curves $t \rightarrow x(t) \in R^n$.
- $R^n$ is called phase space.
- If $n = 2$, $R^2$ is the phase plane.
- If $n = 1$, $R^1$ is the phase line.

Uniqueness in Phase Space

Two solution curves in phase space for an autonomous system cannot meet at a point unless the solution curves coincide.

- If $n = 2$, two solution curves in the phase plane cannot cross, or even touch.
- If the system is not autonomous, solution curves in the phase plane can cross.
Equilibrium Points & Solutions

$x' = f(x)$.

- The system is autonomous.
- $x_0$ is an equilibrium point if $f(x_0) = 0$.
- $x(t) = x_0$ is the corresponding equilibrium solution.
- In phase space, an equilibrium solution plots as a point.
- Nullclines — where one component of the right-hand side $f(x)$ vanishes.

Example

$x' = x^2 - y$
$y' = x - xy$

- $x$-nullcline: $x^2 - y = 0$.
- $y$-nullcline: $x(1 - y) = 0$.
- 3 equilibrium points: $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

Linear Systems

A system is linear if the unknown functions appear linearly in the right-hand sides.

- Appear linearly means that there are no products, powers, or higher order functions.
- Examples
  - SIR is nonlinear.
  - Previous example is linear.
Planar Linear Systems

A planar linear system is one of the form

\[ x' = a(t)x + b(t)y + f(t) \]
\[ y' = c(t)x + d(t)y + g(t) \]

- The coefficients can depend on \( t \).

General Linear Systems

\[
\begin{align*}
x'_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \\
x'_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \\
&\vdots \\
x'_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n
\end{align*}
\]

- The coefficients can depend on \( t \).

- Set

\[
\begin{align*}
x &= (x_1, x_2, \ldots, x_n)^T \\
f(t) &= (f_1(t), f_2(t), \ldots, f_n(t))^T \\
A &= \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\end{align*}
\]

- The system becomes \( x' = Ax + f \).
Existence & Uniqueness

Theorem: Suppose the matrix-valued function $A = A(t)$ and the vector-valued function $f(t)$ are defined and continuous in an interval $I = (\alpha, \beta)$. Then for any $t_0$ in $I$ and any $x_0$ in $\mathbb{R}^n$, the initial value problem

$$x' = Ax + f \text{ with } x(t_0) = x_0$$

has a unique solution defined for all $t$ in $I$.

Homogeneous Systems

A homogeneous system is one of the form

$$x' = Ax$$

Proposition: Suppose that $x_1(t)$, $x_2(t)$, $\ldots$, and $x_k(t)$ are solutions to the homogeneous system, and $c_1$, $c_2$, $\ldots$, and $c_k$ are scalars. Then

$$x(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_kx_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.