Math 211

Lecture #24
Linear Systems of ODEs

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General Linear Systems

\[ x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \]
\[ x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \]
\[ \vdots \]
\[ x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n \]

- The coefficients can depend on \( t \).

Set

\[ x = (x_1, x_2, \ldots, x_n)^T \]
\[ f(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T \]
\[ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \]

- The system becomes \( x' = Ax + f \).
Homogeneous Systems

An homogeneous system is one of the form

\[ x' = Ax \]

Proposition: Suppose that \( x_1(t), x_2(t), \ldots, x_k(t) \) are solutions to the homogeneous system \( x' = Ax \), and \( c_1, c_2, \ldots, c_k \) are scalars. Then

\[ x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_k x_k(t) \]

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

Very Important Example

- The system

\[ x' = Ax \quad \text{with} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

has solutions

\[ x_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad x_2(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \].

- Verify by direct substitution.
- Proposition \( \Rightarrow x(t) = C_1 x_1(t) + C_2 x_2(t) \) is a solution for any constants \( C_1 \) and \( C_2 \).

- Let \( y \) be a solution of \( y' = Ay \). Can we find \( C_1 \) and \( C_2 \) so that

\[ y(t) = C_1 x_1(t) + C_2 x_2(t) \quad \text{for all} \ t? \]

- Let’s ask a simpler question. Can we find \( C_1 \) and \( C_2 \) so that

\[ y(0) = C_1 x_1(0) + C_2 x_2(0)? \]

- Yes, since \( x_1(0) \) and \( x_2(0) \) are linearly independent.
• Uniqueness theorem ⇒
\[ y(t) = C_1x_1(t) + C_2x_2(t) \text{ for all } t. \]
• Thus, every solution to \( x' = Ax \) is a linear combination of \( x_1 \) and \( x_2 \).
  • The general solution to the system is a linear combination of the two solutions \( x_1 \) and \( x_2 \).
• Can we generalize this result?

Key Point in the Argument
• Need to solve the equation
\[ y_0 = C_1x_1(0) + C_2x_2(0) \]
for any \( y_0 = y(0) \).
• Possible if \( x_1(0) \) and \( x_2(0) \) are linearly independent.
• Uniqueness then implies that
\[ y(t) = C_1x_1(t) + C_2x_2(t) \text{ for all } t. \]
• We needed \( x_1(t) \) and \( x_2(t) \) to be linearly independent at only one point.

Proposition: \( x_1(t), x_2(t), \ldots, \) and \( x_k(t) \) solutions to the homogeneous system \( x' = Ax \) on the interval \( I \).
1. If \( x_1(t_0), x_2(t_0), \ldots, \) and \( x_k(t_0) \) are linearly independent for some \( t_0 \in I \), then they are linearly independent for all \( t \in I \).
2. If \( x_1(t_0), x_2(t_0), \ldots, \) and \( x_k(t_0) \) are linearly dependent for some \( t_0 \in I \), then they are linearly dependent for all \( t \in I \).
Linear Independence

Definition: A set of \( k \) solutions to the linear system 
\[ x' = Ax \]
is linearly independent if they are linearly independent at one value of \( t \).

- Proposition \( \Rightarrow \) the solutions are linearly independent for all values of \( t \).

Structure of the Solution Space

Theorem: Suppose that \( x_1(t), x_2(t), \ldots, \) and \( x_n(t) \) are linearly independent solutions to the \( n \times n \) homogeneous system 
\[ x' = Ax \]
on the interval \( I \). Then every solution is a linear combination of \( x_1(t), x_2(t), \ldots, \) and \( x_n(t) \).

- That is, if \( x(t) \) is any solution, then there are constants \( C_1, C_2, \ldots, \) and \( C_n \) such that
\[ x(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t). \]

- The general solution is a linear combination of \( x_1(t), \)
\( x_2(t), \ldots, \) and \( x_n(t) \).

Solution Strategy

- The obvious strategy for completely solving an \( n \times n \) system is to look for \( n \) linearly independent solutions.

Definition: A set of \( n \) linear independent solutions to the \( n \times n \) homogeneous system 
\[ x' = Ax \]
is called a fundamental set of solutions.

- We will look for fundamental sets of solutions.
Example 1: $x' = Ax$

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$x_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ and $x_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

are a fundamental set of solutions.

Example 2: $x' = Ax$

$A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$

$x_1(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $x_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

are a fundamental set of solutions.

Linear Systems with Constant Coefficients

• We will solve homogeneous equations first.
• We will be able to find explicit solutions.
• To motivate what we do, we will start with the easiest case, dimension = 1.
  • One equation: $x' = ax$
    • $a$ is a constant.
  • Solution: $x(t) = Ce^{at}$
• All solutions are exponentials. Can we find exponential solutions to a system of equations?
Exponential Solutions to $x' = Ax$

- Look for solution of the form $x(t) = e^{\lambda t}v$, where $v$ is a vector with constant entries.
- Substituting we get
  
  $x' = \lambda e^{\lambda t}v$
  
  $Ax = e^{\lambda t}Av$

- Hence $x' = Ax \iff Av = \lambda v$
- If $Av = \lambda v$ then $x(t) = e^{\lambda t}v$ is a solution.
- Can we find $\lambda$ and $v$ such that $Av = \lambda v$?

Eigenvalues & Eigenvectors

Definition: $\lambda$ is an eigenvalue of $A$ if there is a nonzero vector $v$ such that $Av = \lambda v$. If $\lambda$ is an eigenvalue of $A$, then any vector $v$ such that $Av = \lambda v$ is called an eigenvector associated with $\lambda$.

- If $\lambda$ an eigenvalue of $A$, and $v$ is an associated nonzero eigenvector, then $x(t) = e^{\lambda t}v$ is a solution to $x' = Ax$.
- How do we find eigenvalues and eigenvectors?

Finding Eigenvalues

$\lambda$ is an eigenvalue of $A$

$\iff$ there is a vector $v \neq 0$ such that $Av = \lambda v$.

$\iff v \neq 0$ and $0 = Av - \lambda v$

$= Av - \lambda Iv$

$= (A - \lambda I)v$

$\iff A - \lambda I$ has a nontrivial nullspace.

$\iff \det(A - \lambda I) = 0.$
Example

\[ A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

\[ A - \lambda I = \begin{pmatrix} -4 - \lambda & 2 \\ -3 & 1 - \lambda \end{pmatrix} \]

\[ \det(A - \lambda I) = (-4 - \lambda)(1 - \lambda) + 6 \]

\[ = \lambda^2 + 3\lambda + 2 \]

\[ = (\lambda + 1)(\lambda + 2) \]

- \( A \) has eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \).
• Usually \( p(\lambda) = \det(A - \lambda I) = 0 \) has \( n \) roots. Usually \( A \) has \( n \) eigenvalues.

• Each eigenvalue \( \lambda \) has by definition an associated nonzero eigenvector \( v \).

• Each nonzero eigenvector leads to a solution, \( x(t) = e^{\lambda t}v \).

• There is at least one solution for every eigenvalue, so we expect \( n \) different solutions.

• Are they linearly independent?

Finding Eigenvectors

• \( v \) is an eigenvector associated with the eigenvalue \( \lambda \) if

\[
A v = \lambda v
\]

\[
\Leftrightarrow (A - \lambda I)v = 0
\]

\[
\Leftrightarrow v \in \text{null}(A - \lambda I)
\]

• The set of all eigenvectors associated to the eigenvalue \( \lambda \) is equal to the nullspace of \( A - \lambda I \). It is a subspace of \( \mathbb{R}^n \) called the eigenspace of \( \lambda \).

Example

\[
A = \begin{pmatrix}
-4 & 2 \\
-3 & 1
\end{pmatrix}
\]

• \( A \) has eigenvalues

\[
\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2.
\]

• Look for associated eigenvectors. Each leads to a solution to \( x' = Ax \).
\[ \lambda_1 = -1 \]
\[ A - \lambda_1 I = \begin{pmatrix} -4 & 1 \\ -3 & 1 + 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix} \]
\[ v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ is an eigenvector} \]
\[ x_1(t) = e^{\lambda_1 t} v_1 = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ is a solution.} \]

\[ \lambda_2 = -2 \]
\[ A - \lambda_2 I = \begin{pmatrix} -4 & 2 \\ -3 & 1 + 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} \]
\[ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector} \]
\[ x_2(t) = e^{\lambda_2 t} v_2 = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is a solution.} \]

\[ x' = Ax \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]
- The system has solutions
  \[ x_1(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad x_2(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]
- \( x_1(0) = v_1 \) and \( x_2(0) = v_2 \) are linearly independent.
- \( x_1 \) and \( x_2 \) form a fundamental set of solutions.
\[ x' = Ax \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

- The general solution is the set of all linear combinations:
  \[ x(t) = C_1x_1(t) + C_2x_2(t) \]
  \[ = C_1e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
  \[ = \begin{pmatrix} 2C_1e^{-t} + C_2e^{-2t} \\ 3C_1e^{-t} + C_2e^{-2t} \end{pmatrix} \]

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**Procedure to Solve** \( x' = Ax \)

- Find the eigenvalues of \( A \)
- the roots of \( \det(A - \lambda I) = 0 \)
- For each eigenvalue \( \lambda \) find the eigenspace
  - \( \text{null}(A - \lambda I) \)
- If \( \lambda \) is an eigenvalue and \( v \) is an associated nonzero eigenvector, \( x(t) = e^{\lambda t}v \) is a solution.
- Show that \( n \) of these are linearly independent, if we can.