Math 211

Lecture #24

Linear Systems of ODEs

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General Linear Systems

\[ x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \]
\[ x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \]
\[ \vdots = \vdots \]
\[ x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n \]

- The coefficients can depend on \( t \).
Set

\[ x = (x_1, x_2, \ldots, x_n)^T \]

\[ f(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T \]

\[ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \]

The system becomes \( x' = Ax + f. \)
Homogeneous Systems

An *homogeneous* system is one of the form

\[ x' = Ax \]

**Proposition:** Suppose that \( x_1(t), x_2(t), \ldots, \) and \( x_k(t) \) are solutions to the homogeneous system \( x' = Ax \), and \( c_1, c_2, \ldots, \) and \( c_k \) are scalars. Then

\[ x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_k x_k(t) \]

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.
Very Important Example

- The system

\[ x' = Ax \quad \text{with} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

has solutions

\[ x_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad x_2(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} . \]

- Verify by direct substitution.

- Proposition \( \Rightarrow x(t) = C_1 x_1(t) + C_2 x_2(t) \) is a solution for any constants \( C_1 \) and \( C_2 \).
• Let $y$ be a solution of $y' = Ay$. Can we find $C_1$ and $C_2$ so that

$$y(t) = C_1 x_1(t) + C_2 x_2(t) \quad \text{for all } t?$$

• Let’s ask a simpler question. Can we find $C_1$ and $C_2$ so that

$$y(0) = C_1 x_1(0) + C_2 x_2(0)?$$

♦ Yes, since $x_1(0)$ and $x_2(0)$ are linearly independent.
• Uniqueness theorem \( \Rightarrow \)

\[ y(t) = C_1 x_1(t) + C_2 x_2(t) \quad \text{for all } t. \]

• Thus, every solution to \( x' = Ax \) is a linear combination of \( x_1 \) and \( x_2 \).

♦ The general solution to the system is a linear combination of the two solutions \( x_1 \) and \( x_2 \).

• Can we generalize this result?
Key Point in the Argument

- Need to solve the equation

\[ y_0 = C_1x_1(0) + C_2x_2(0) \]

for any \( y_0 = y(0) \).

- Possible if \( x_1(0) \) and \( x_2(0) \) are linearly independent.

- Uniqueness then implies that

\[ y(t) = C_1x_1(t) + C_2x_2(t) \quad \text{for all } t \]

- We needed \( x_1(t) \) and \( x_2(t) \) to be linearly independent at only one point.
Proposition: \( x_1(t), x_2(t), \ldots, \) and \( x_k(t) \) solutions to the homogeneous system \( x' = Ax \) on the interval \( I \).

1. If \( x_1(t_0), x_2(t_0), \ldots, \) and \( x_k(t_0) \) are linearly independent for some \( t_0 \in I \), then they are linearly independent for all \( t \in I \).

2. If \( x_1(t_0), x_2(t_0), \ldots, \) and \( x_k(t_0) \) are linearly dependent for some \( t_0 \in I \), then they are linearly dependent for all \( t \in I \).
Linear Independence

**Definition:** A set of $k$ solutions to the linear system $x' = Ax$ is *linearly independent* if they are linearly independent at one value of $t$.

- **Proposition** $\Rightarrow$ the solutions are linearly independent for all values of $t$. 
Structure of the Solution Space

**Theorem:** Suppose that $x_1(t), x_2(t), \ldots, \text{and } x_n(t)$ are *linearly independent* solutions to the $n \times n$ homogeneous system $x' = Ax$ on the interval $I$. Then every solution is a linear combination of $x_1(t), x_2(t), \ldots, \text{and } x_n(t)$.

- That is, if $x(t)$ is any solution, then there are constants $C_1, C_2, \ldots, \text{and } C_n$ such that

\[
x(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t).
\]

- The general solution is a linear combination of $x_1(t), x_2(t), \ldots, \text{and } x_n(t)$. 


Solution Strategy

- The obvious strategy for completely solving an $n \times n$ system is to look for $n$ linearly independent solutions.

**Definition:** A set of $n$ linear independent solutions to the $n \times n$ homogeneous system $x' = Ax$ is called a *fundamental set of solutions*.

- We will look for fundamental sets of solutions.
Example 1: $\mathbf{x}' = A\mathbf{x}$

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$x_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ and $x_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

are a fundamental set of solutions.
Example 2: \( x' = Ax \)

\[
A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}
\]

\[
x_1(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad x_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]

are a fundamental set of solutions.
Linear Systems with Constant Coefficients

- We will solve \textit{homogeneous equations} first.
- We will be able to find explicit solutions.
- To motivate what we do, we will start with the easiest case, dimension $= 1$.
  - One equation: $x' = ax$
    - $a$ is a constant.
  - Solution: $x(t) = Ce^{at}$
- All solutions are exponentials. Can we find exponential solutions to a system of equations?
Exponential Solutions to $x' = Ax$

- Look for solution of the form $x(t) = e^{\lambda t} \mathbf{v}$, where $\mathbf{v}$ is a vector with constant entries.

- Substituting we get

  $$x' = \lambda e^{\lambda t} \mathbf{v}$$

  $$Ax = e^{\lambda t} A\mathbf{v}$$

- Hence $x' = Ax \iff A\mathbf{v} = \lambda \mathbf{v}$

- If $A\mathbf{v} = \lambda \mathbf{v}$ then $x(t) = e^{\lambda t} \mathbf{v}$ is a solution.

- Can we find $\lambda$ and $\mathbf{v}$ such that $A\mathbf{v} = \lambda \mathbf{v}$?
Eigenvalues & Eigenvectors

**Definition:** $\lambda$ is an *eigenvalue* of $A$ if there is a nonzero vector $v$ such that $Av = \lambda v$. If $\lambda$ is an eigenvalue of $A$, then any vector $v$ such that $Av = \lambda v$ is called an *eigenvector associated with* $\lambda$.

- If $\lambda$ an eigenvalue of $A$, and $v$ is an associated nonzero eigenvector, then $x(t) = e^{\lambda t}v$ is a solution to $x' = Ax$.
- How do we find eigenvalues and eigenvectors?
Finding Eigenvalues

\( \lambda \) is an eigenvalue of \( A \)

\( \iff \) there is a vector \( \mathbf{v} \neq \mathbf{0} \) such that \( A\mathbf{v} = \lambda \mathbf{v} \).

\( \iff \mathbf{v} \neq \mathbf{0} \) and \( \mathbf{0} = A\mathbf{v} - \lambda \mathbf{v} \)

\[ = A\mathbf{v} - \lambda I \mathbf{v} \]

\[ = (A - \lambda I) \mathbf{v} \]

\( \iff A - \lambda I \) has a nontrivial nullspace.

\( \iff \det(A - \lambda I) = 0. \)
Example

\[
A = \begin{pmatrix}
-4 & 2 \\
-3 & 1 \\
\end{pmatrix}
\]

\[
A - \lambda I = \begin{pmatrix}
-4 - \lambda & 2 \\
-3 & 1 - \lambda \\
\end{pmatrix}
\]

\[
\det(A - \lambda I) = (-4 - \lambda)(1 - \lambda) + 6
\]

\[
= \lambda^2 + 3\lambda + 2
\]

\[
= (\lambda + 1)(\lambda + 2)
\]

- \(A\) has eigenvalues \(\lambda_1 = -1\) and \(\lambda_2 = -2\).
Finding eigenvalues

\[
\text{det}(A - \lambda I)
\]

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

\[
A - \lambda I = \begin{pmatrix}
  a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda
\end{pmatrix}
\]
• If $A$ is an $n \times n$ matrix $p(\lambda) = \text{det}(A - \lambda I)$ is a polynomial of degree $n$.

**Definition:** The *characteristic polynomial* of the $n \times n$ matrix $A$ is

$$p(\lambda) = \text{det}(A - \lambda I).$$

The *characteristic equation* is $p(\lambda) = 0$.

• The *eigenvalues* of $A$ are the roots of the characteristic equation.
• Usually $p(\lambda) = \det(A - \lambda I) = 0$ has $n$ roots. Usually $A$ has $n$ eigenvalues.

• Each eigenvalue $\lambda$ has by definition an associated nonzero eigenvector $\mathbf{v}$.

• Each nonzero eigenvector leads to a solution, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.

• There is at least one solution for every eigenvalue, so we expect $n$ different solutions.

  ♦ Are they linearly independent?
Finding Eigenvectors

• $v$ is an eigenvector associated with the eigenvalue $\lambda$ if

\[ Av = \lambda v \]
\[ \iff (A - \lambda I)v = 0 \]
\[ \iff v \in \text{null}(A - \lambda I) \]

• The set of all eigenvectors associated to the eigenvalue $\lambda$ is equal to the nullspace of $A - \lambda I$. It is a subspace of $\mathbb{R}^n$ called the eigenspace of $\lambda$. 
Example

\[ A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

- \( A \) has eigenvalues
  \[ \lambda_1 = -1 \] and \( \lambda_2 = -2. \)

- Look for associated eigenvectors. Each leads to a solution to \( x' = Ax \).
\[ \lambda_1 = -1 \]

\[ A - \lambda_1 I = \begin{pmatrix} -4 & 1 & 2 \\ -3 & 1 & 1 + 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix} \]

\[ \mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \] is an eigenvector

\[ \mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \] is a solution.
\[ \lambda_2 = -2 \]

\[ A - \lambda_2 I = \begin{pmatrix} -4 + 2 & 2 \\ -3 & 1 + 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} \]

\[ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector} \]

\[ \mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2 = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is a solution.} \]
\( x' = Ax \) where \( A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \)

- The system has solutions

\[
x_1(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad x_2(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

- \( x_1(0) = v_1 \) and \( x_2(0) = v_2 \) are linearly independent.

- \( x_1 \) and \( x_2 \) form a fundamental set of solutions.
\[ x' = Ax \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

- The general solution is the set of all linear combinations:

\[ x(t) = C_1 x_1(t) + C_2 x_2(t) \]

\[ = C_1 e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 2C_1 e^{-t} + C_2 e^{-2t} \\ 3C_1 e^{-t} + C_2 e^{-2t} \end{pmatrix} \]
Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of $A$
  - the roots of $\det(A - \lambda I) = 0$
- For each eigenvalue $\lambda$ find the eigenspace
  - $= \text{null}(A - \lambda I)$
- If $\lambda$ is an eigenvalue and $\mathbf{v}$ is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that $n$ of these are linearly independent, if we can.