Math 211

Lecture #25

Exponential Solutions

October 26, 2001
Homogeneous Systems

- These are systems of the form

\[ x' = Ax, \]

where \( A \) is an \( n \times n \) matrix.

- We are looking primarily at homogeneous systems with constant coefficients.
Structure of the Solution Space

**Theorem:** Suppose that $x_1(t), x_2(t), \ldots, x_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $x' = Ax$ on the interval $I$. Then every solution is a linear combination of $x_1(t), x_2(t), \ldots, x_n(t)$.

That is, if $x(t)$ is a solution, then there are constants $C_1, C_2, \ldots, C_n$ such that

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots + C_n x_n(t).$$
Solution Strategy

- The obvious strategy for completely solving the system is to look for \( n \) linearly independent solutions.

**Definition:** A set of \( n \) linear independent solutions to the \( n \times n \) homogeneous system \( x' = Ax \) is called a *fundamental set of solutions*.

- We will look for fundamental sets of solutions.
Procedure to Solve $x' = Ax$

- Find the eigenvalues of $A$
  - the roots of $p(\lambda) = \det(A - \lambda I) = 0$
- For each eigenvalue $\lambda$ find the eigenspace
  - $\lambda \text{null}(A - \lambda I)$
- If $\lambda$ is an eigenvalue and $v$ is an associated eigenvector, $x(t) = e^{\lambda t}v$ is a solution.
- Hope that $n$ of these are linearly independent.
Example

\[ x' = Ax \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

• \( A \) has eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \).

• Look for associated eigenvectors.

\( \lambda_1 = -1 \). Eigenvector: \( v_1 = (2, 3)^T \).

▶ Solution: \( x_1(t) = e^{\lambda_1 t}v_1 = e^{-t}(2, 3)^T \).

\( \lambda_2 = -2 \). Eigenvector: \( v_2 = (1, 1)^T \).

▶ Solution: \( x_2(t) = e^{\lambda_2 t}v_2 = e^{-2t}(1, 1)^T \).
• $x_1(0) = v_1 = (2, 3)^T$ and $x_2(0) = v_2 = (1, 1)^T$ are linearly independent.

• $x_1$ and $x_2$ form a fundamental set of solutions.

• The general solution is the set of all linear combinations:

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

$$= C_1 e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
Solving $x' = Ax$

Cases to be Considered

- Distinct real eigenvalues.
  - In this case the method works as described.
- Complex eigenvalues.
  - The method yields complex solutions, but we will want real solutions.
- Repeated eigenvalues.
  - The method does not always give enough solutions.
    - This is the hard case.
Planar System $x' = Ax$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

In nonvector form

$$x'_1 = a_{11}x_1 + a_{12}x_2$$

$$x'_2 = a_{21}x_1 + a_{22}x_2$$
The Characteristic Polynomial

\[ p(\lambda) = \det(A - \lambda I) \]

\[ = \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \]

\[ = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \]

\[ = \lambda^2 - T\lambda + D, \]

where

- \( D = a_{11}a_{22} - a_{12}a_{21} = \det(A) \)
- \( T = a_{11} + a_{22} = \text{tr}(A) \) is the trace of \( A \).

- The trace of a matrix is the sum of its diagonal elements.
The Eigenvalues of $A$

- The eigenvalues of $A$ are the roots of the characteristic equation $p(\lambda) = \lambda^2 - T\lambda + D = 0$.

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$ 

- Three cases:
  - 2 distinct real roots if $T^2 - 4D > 0$
  - 2 complex conjugate roots if $T^2 - 4D < 0$
  - Double real root if $T^2 - 4D = 0$
Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

**Proposition:** Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix $A$, and that $v_1 \neq 0$ and $v_2 \neq 0$ are eigenvectors associated with $\lambda_1$ and $\lambda_2$, respectively. Then $v_1$ and $v_2$ are linearly independent.
Two Distinct Real Eigenvalues

\[ \lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2} \]

- \( T^2 - 4D > 0 \) so \( \lambda_1 < \lambda_2 \).
- There are associated nonzero eigenvectors \( v_1 \) and \( v_2 \).
- Solutions \( x_1(t) = e^{\lambda_1 t}v_1 \) and \( x_2(t) = e^{\lambda_2 t}v_2 \).
- \( x_1(0) = v_1 \) and \( x_2(0) = v_2 \) are linearly independent; \( x_1(t) \) and \( x_2(t) \) form a fundamental set of solutions.
- The general solution is \( x(t) = C_1 e^{\lambda_1 t}v_1 + C_2 e^{\lambda_2 t}v_2 \).
Example

\[ x' = Ax \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix} \]

- **Characteristic polynomial:** \( p(\lambda) = \lambda^2 - 4 \).
- **Eigenvalues:** \( \lambda_1 = -2 \) and \( \lambda_2 = 2 \).
  - \( \lambda_1 = -2 \). Eigenvector: \( v_1 = (-2, 1)^T \).
    - Solution: \( x_1(t) = e^{\lambda_1 t} v_1 = e^{-2t}(-2, 1)^T \).
  - \( \lambda_2 = 2 \). Eigenvector: \( v_2 = (-1, 1)^T \).
    - Solution: \( x_2(t) = e^{\lambda_2 t} v_2 = e^{2t}(-1, 1)^T \).
• $x_1$ and $x_2$ are a fundamental set of solutions.

• The general solution is

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

$$= C_1 e^{-2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$
Initial Value Problem

Solve $x' = Ax$ with the initial condition $x(0) = (1, 4)^T$.

• We need

$$x(0) = C_1 x_1(0) + C_2 x_2(0)$$

♦ $C_1 = -5$ and $C_2 = 9$.

• The solution is

$$x(t) = -5x_1(t) + 9x_2(t)$$

$$= \begin{pmatrix} 10e^{-2t} - 9e^{2t} \\ -5e^{-2t} + 9e^{2t} \end{pmatrix}.$$
Homogeneous Systems

\[ x' = Ax \]

**Proposition:** Suppose that \( x_1(t), x_2(t), \ldots, \) and \( x_k(t) \) are solutions to the homogeneous system, and \( c_1, c_2, \ldots, \) and \( c_k \) are scalars. Then

\[ x(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_kx_k(t) \]

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.
**Linear Independence**

**Definition:** A set of $k$ solutions to the linear system $x' = Ax$ is *linearly independent* if they are linearly independent at one value of $t$.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of $t$. 

Eigenvalues & Eigenvectors

**Definition:** \( \lambda \) is an *eigenvalue* of \( A \) if there is a nonzero vector \( v \) such that \( Av = \lambda v \). If \( \lambda \) is an eigenvalue of \( A \), then any vector \( v \) such that \( Av = \lambda v \) is called an *eigenvector associated with* \( \lambda \).

- \( \lambda \) an eigenvalue of \( A \), \( v \) an associated eigenvector
  \( \Rightarrow \) \( x(t) = e^{\lambda t} v \) is a solution to \( x' = Ax \).

- The set of eigenvectors associated with the eigenvalue \( \lambda \) is the subspace \( \text{null}(A - \lambda I) \), and is called the eigenspace of \( \lambda \).