Math 211

Lecture #27
Planar Systems

October 31, 2001

Procedure to Solve $x' = Ax$

- Find the eigenvalues of $A$
  - the roots of $p(\lambda) = \det(A - \lambda I) = 0$
- For each eigenvalue $\lambda$ find the eigenspace
  - $\text{null}(A - \lambda I)$
- If $\lambda$ is an eigenvalue and $v$ is an associated eigenvector, $x(t) = e^{\lambda t}v$ is a solution.
- Hope that $n$ of these are linearly independent.

Planar System $x' = Ax$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

- The characteristic polynomial is
  $$p(\lambda) = \lambda^2 - T\lambda + D.$$  
  where $T = \text{tr} \, A$ and $D = \det A$.
• The eigenvalues of $A$ are the roots of
  \[ p(\lambda) = \lambda^2 - T\lambda + D, \]
  \[ \lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}. \]
• Three cases:
  • 2 distinct real roots if $T^2 - 4D > 0$
  • 2 complex conjugate roots if $T^2 - 4D < 0$
  • Double real root if $T^2 - 4D = 0$

Complex Eigenpairs

$A$ a real matrix
• Complex eigenvalues come in conjugate pairs $\lambda$ and $\overline{\lambda}$.
• The associated eigenvectors also come in conjugate pairs $w$ and $\overline{w}$.
• $\lambda \neq \overline{\lambda} \implies w$ and $\overline{w}$ are linearly independent.
• We get complex exponential solutions
  \[ z(t) = e^{\lambda t}w \text{ and } \overline{z}(t) = e^{\overline{\lambda} t} \overline{w}. \]
• $z$ and $\overline{z}$ are linearly independent complex valued solutions to $x' = Ax$.

Complex Eigenpairs – Real Solutions

\[ z(t) = x(t) + iy(t) \quad \text{and} \quad \overline{z}(t) = x(t) - iy(t) \]

\[ x(t) = \text{Re}(z(t)) = \frac{z(t) + \overline{z}(t)}{2} \]

\[ y(t) = \text{Im}(z(t)) = \frac{z(t) - \overline{z}(t)}{2i} \]

• $x(t)$ and $y(t)$ are real valued solutions.
• $x(t)$ and $y(t)$ are linearly independent.
Example

\[ x' = Ax \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}. \]

• Complex Solutions

\[ z(t) = e^{\lambda t} w = e^{(1+2i)t} \begin{pmatrix} 3 - i \\ 1 \end{pmatrix} \]

\[ \pi(t) = e^{\lambda t} \overline{w} = e^{(1-2i)t} \begin{pmatrix} 3 + i \\ 1 \end{pmatrix} \]

• \( z \) and \( \pi \) form a complex valued fundamental set of solutions.

• Real Solutions

\[ x(t) = \text{Re}(z(t)) = e^{t} \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \]

\[ y(t) = \text{Im}(z(t)) = e^{t} \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} \]

• \( x \) and \( y \) form a real-valued fundamental set of solutions.

Initial Value Problem

Solve \( x' = Ax \& x(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \), where \( A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix} \).

The solution is

\[ u(t) = 3e^{t} \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} 
+ 4e^{t} \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} 
= e^{t} \begin{pmatrix} 5 \cos 2t + 15 \sin 2t \\ 3 \cos 2t + 4 \sin 2t \end{pmatrix} \]
Summary — Complex Eigenvalues

Suppose $A$ is a real $2 \times 2$ matrix with

- complex conjugate eigenvalues $\lambda$ and $\bar{\lambda}$, and
- associated nonzero eigenvectors $w$ and $\bar{w}$.

Then

- $z(t) = e^{\lambda t}w$ and $\bar{z}(t) = e^{\bar{\lambda} t}\bar{w}$ form a complex valued fundamental set of solutions, and
- $x(t) = \text{Re}(z(t))$ and $y(t) = \text{Im}(z(t))$ form a real valued fundamental set of solutions.

Examples

$x' = Ax$

where

- $A = \begin{pmatrix} 7 & 30 \\ -3 & -11 \end{pmatrix}$
- $A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$

Double Real Root

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T}{2}.$$ 

- $T^2 - 4D = 0$

- First possibility:
  - Eigenspace has dimension 2: $\implies A = \lambda I$.
  - Every vector is an eigenvector. Every solution has the form $x(t) = e^{\lambda t}v$. 

Example

\[ x' = Ax \quad \text{where} \quad A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix} \]

- \[ p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2; \quad \lambda = -2 \]
- \[ A - \lambda I = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}; \quad v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \]
- Eigenspace has dimension 1, with basis \( v_1 \).
- One solution:
  \[ x_1(t) = e^{\lambda t}v_1 = e^{-2t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}. \]

Double Real Root

- Suppose the eigenspace has dimension 1.
- Spanned by the eigenvector \( v_1 \neq 0 \)
- Standard procedure gives only one solution,
  \[ x_1(t) = e^{\lambda t}v_1. \]
- Look for a second solution of the form
  \[ x_2(t) = e^{\lambda t}[v_2 + tv_1] \]

- Second solution of the form
  \[ x_2(t) = e^{\lambda t}[v_2 + tv_1] \]
  \[ x_2' = e^{\lambda t}[(v_1 + \lambda v_2) + \lambda v_1] \]
  \[ Ax_2 = e^{\lambda t}[Av_2 + tAv_1] \]
- \[ x_2' = Ax_2 \Leftrightarrow \]
  \[ Av_1 = \lambda v_1 \quad \text{and} \quad Av_2 = v_1 + \lambda v_2. \]

- Need:
  - \( v_1 \) to be an eigenvector.
  - \( (A - \lambda I)v_2 = v_1. \)
Procedure in Degenerate Planar Case

- Find the (only) eigenvalue $\lambda_1$.
- Find an eigenvector $v_1 \neq 0$.
- Find $v_2$ with $(A - \lambda I)v_2 = v_1$.
  - Start with any vector $w$ not a multiple of $v_1$
  - $(A - \lambda I)w = \alpha v_1$ with $\alpha \neq 0$.
  - Set $v_2 = \frac{1}{\alpha}w$. $v_2$ is not a multiple of $v_1$.
- $x_1(t) = e^{\lambda t}v_1$ and $x_2(t) = e^{\lambda t}[v_2 + tv_1]$ form a fundamental set of solutions.

Example (cont.)

- Start with $w = (1, 0)^T$.
- $v_2 = -w = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
- Fundamental set of solutions:
  - $x_1(t) = e^{\lambda t}v_1 = e^{-2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
  - $x_2(t) = e^{\lambda t}[v_2 + tv_1]$
    - $= e^{-2t} \begin{pmatrix} -1 - 3t \\ t \end{pmatrix}$.

Examples

Solve $x' = Ax$, where

- $A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$

- $A = \begin{pmatrix} 0 & 9 \\ -1 & -6 \end{pmatrix}$
Planar System \( \mathbf{x}' = A\mathbf{x} \)
- Equilibrium points for the system
- Set of equilibrium points equals \( \text{null}(A) \).
- \( A \) nonsingular \( \Rightarrow \) only equilibrium point is \( \mathbf{0} \).
- Can we list the types of all possible equilibrium points for planar linear systems?
  - Six most important cases.
  - Look at solution curves in the phase plane.

Exponential Solutions
\( x(t) = Ce^{\lambda t}v \)
- The solution curve is a straight half-line through \( C\mathbf{v} \).
  - Sometimes called half-line solutions.
- If \( \lambda > 0 \) the solution starts at \( \mathbf{0} \) for \( t = -\infty \), and tends to \( \infty \) as \( t \to \infty \). \textit{Unstable solution}
- If \( \lambda < 0 \) the solution starts at \( \infty \) for \( t = -\infty \), and tends to \( \mathbf{0} \) as \( t \to \infty \). \textit{Stable solution}

Distinct Real Eigenvalues
- \( p(\lambda) = \lambda^2 - T\lambda + D \) with \( T^2 - 4D > 0 \).
  \[ \lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2} \]
- Eigenvectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). General solution
  \[ x(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 \]
Saddle Point

- $\lambda_1 < 0 < \lambda_2$
- General solution $x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$
- Two stable exponential solutions ($C_2 = 0$)
- Two unstable exponential solutions ($C_1 = 0$).
- $C_1 \neq 0$ and $C_2 \neq 0$.
  - As $t \to \infty$, $x(t) \to \infty$, approaching the half-line through $C_2 v_2$.
  - As $t \to -\infty$, $x(t) \to \infty$, approaching the half-line through $C_2 v_1$.

Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution $x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$
- Four stable exponential solutions.
- All solutions $\to 0$ as $t \to \infty$. (Stable)
  - Tangent to $C_2 v_2$ if $C_2 \neq 0$.
- All solutions $\to \infty$ as $t \to -\infty$.
  - $\parallel$ to the half line through $C_1 v_1$ if $C_1 \neq 0$.

Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution $x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$
- Four unstable exponential solutions.
- All solutions $\to 0$ as $t \to -\infty$.
  - Tangent to $C_1 v_1$ if $C_1 \neq 0$.
- All solutions $\to \infty$ as $t \to \infty$. (Unstable)
  - $\parallel$ to the half line through $C_2 v_2$ if $C_2 \neq 0$. 
Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

**Proposition:** Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix $A$, and that $v_1 \neq 0$ and $v_2 \neq 0$ are eigenvectors associated with $\lambda_1$ and $\lambda_2$, respectively. Then $v_1$ and $v_2$ are linearly independent.